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Brini, Andrea; Borot, Gaetan

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# Chern–Simons theory on spherical Seifert manifolds, topological strings and integrable systems

Gaëtan Borot<sup>1</sup> and Andrea Brini<sup>2</sup>

## Abstract

We consider the Gopakumar–Ooguri–Vafa correspondence, relating  $U(N)$  Chern–Simons theory at large  $N$  to topological strings, in the context of spherical Seifert 3-manifolds. These are quotients  $\mathbb{S}^\Gamma = \Gamma \backslash \mathbb{S}^3$  of the three-sphere by the free action of a finite isometry group. Guided by string theory dualities, we propose a large  $N$  dual description in terms of both A- and B-twisted topological strings on (in general non-toric) local Calabi–Yau threefolds. The target space of the B-model theory is obtained from the spectral curve of Toda-type integrable systems constructed on the double Bruhat cells of the simply-laced group identified by the ADE label of  $\Gamma$ . Its mirror A-model theory is realized as the local Gromov–Witten theory of suitable ALE fibrations on  $\mathbb{P}^1$ , generalizing the results known for lens spaces. We propose an explicit construction of the family of target manifolds relevant for the correspondence, which we verify through a large  $N$  analysis of the matrix model that expresses the contribution of the trivial flat connection to the Chern–Simons partition function. Mathematically, our results put forward an identification between the  $1/N$  expansion of the  $\mathrm{sl}_{N+1}$  LMO invariant of  $\mathbb{S}^\Gamma$  and a suitably restricted Gromov–Witten/Donaldson–Thomas partition function on the A-model dual Calabi–Yau. This  $1/N$  expansion, as well as that of suitable generating series of perturbative quantum invariants of fiber knots in  $\mathbb{S}^\Gamma$ , is computed by the Eynard–Orantin topological recursion.

## 1 Introduction

In a series of celebrated works [GV99, OV00], Gopakumar, Ooguri and Vafa (GOV) proposed the existence of a duality between  $U(N)$  Chern–Simons theory at level  $k$  on  $\mathbb{S}^3$  [Wit89] and the topological A-model on the resolved conifold  $Y = \mathrm{Tot}[\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1]$ . From a physical perspective, this identification provides a concrete instance, and one where exact computations can be performed in detail, of ’t Hooft’s idea that the  $1/N$  expansion of a gauge theory with adjoint fields in the strong  $g_{\mathrm{YM}}^2 N$  limit should be amenable to a dual description in terms of a first quantized string theory. Originally restricted to the partition function and closed string observables [GV99], the correspondence was later extended to incorporate Wilson loops along the unknot [OV00] and topological branes; progress in open/closed mirror symmetry [HIV00, AV00, BKMP09] has further allowed to rephrase the correspondence in terms of the topological B-model on the

<sup>1</sup>Max Planck Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany. E-mail: gborot@mpim-bonn.mpg.de

<sup>2</sup>UMR 5149 du CNRS, Institut Montpellierain Alexander Grothendieck, Université de Montpellier, case courrier 51, 34095 Montpellier Cedex 5, France. E-mail: andrea.brini@univ-montp2.fr

smoothing of the conifold singularity.

Mathematically, the main consequence of this physics-inspired duality is a striking connection of theories of invariants from two domains of mathematics that are *a priori* quite separated. On the one hand, Witten’s heuristic approach to Chern–Simons invariants can be recast in the context of quantum groups and modular tensor categories to yield *bona fide* invariants of links in 3-manifolds [RT90, RT91]; when the Chern–Simons gauge group is  $U(N)$  or  $SO/Sp(N)$ , this leads respectively to the HOMFLY and Kauffman invariants of links. Furthermore, the perturbative expansion of the Chern–Simons functional integral around the trivial flat connection leads to the theory of finite type invariants [BN06], via the Kontsevich integral and Lê–Murakami–Ohtsuki (LMO) invariants. On the flip side, the topological A-model on a Calabi–Yau 3-fold  $X$  is mathematically defined in terms of suitable theories of moduli of curves in  $X$ , either via stable maps [Kon94] or ideal sheaves [DT98]. In particular, for the case of the unknot the Gopakumar–Ooguri–Vafa correspondence asserts that Chern–Simons knot invariants should be identified with suitable virtual counts of open Riemann surfaces on the dual Calabi–Yau 3-fold  $X$ . By mirror symmetry and the remodeling formalism [BKMP09, EO15], this can be recast in the form of the topological recursion of [EO07] on the mirror curve of  $X$ .

As a detailed instance of the gauge/string correspondence, and because of its far-reaching implications in geometry and topology, the GOV correspondence has been the subject of intense study both in the physics and mathematics communities. After the relation between Gromov–Witten invariants of the resolved conifold and the  $\mathfrak{sl}_{N+1}$  quantum invariant of the unknot in  $S^3$  had been proved [FP03, KL02], a natural question was whether the correspondence could be extended so as to encompass other classical gauge groups [BFM04, BFM05], knots<sup>3</sup> [LMV00, BEM12, DSV13, AENV14], and 3-manifolds. The generalization to manifolds beyond  $S^3$  is perhaps the least studied, with all results to date confined to the case of lens spaces [AKMV04, HY09, BGST10].

## 1.1 Scope of the paper

The purpose of this paper is to propose an extension of the GOV correspondence to the case of spherical Seifert manifolds. Our objects of study will be quotients  $S^\Gamma = \Gamma \backslash S^3$  by the free isometric action of a cyclic or binary polyhedral group  $\Gamma \subset SU(2)$ ; one notable example is the Poincaré homology sphere, corresponding to  $\Gamma = P_{120}$  being the binary icosahedral group. We offer here a conjectural dual description of the  $1/N$  expansion in terms of both A- and B-type topological strings, together with a precision check for the contribution of the trivial flat connection, as follows.

On one hand, we associate to each  $\Gamma$  a local Calabi–Yau 3-fold  $Y^\Gamma$ , serving as the A-model target space; this is constructed in Section 3.1 by a natural  $\Gamma$ -equivariant generalization of the conifold transition of [GV99] for  $T^*S^3$ . When  $\Gamma$  is non-abelian, the  $\Gamma$ -action has the effect of reducing the rank of the automorphism group of  $Y^\Gamma$  to two, so that  $Y^\Gamma$  is non-toric. At first sight this may be a hindrance towards finding a mirror B-model picture, as in particular there is no explicit Hori–

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<sup>3</sup>In an allied context, a vast program of computation of HOMFLY invariants, exploring also possible new relations with matrix models, has recently been undertaken by the mathematical physicists at ITEP; see in particular [AMMM14, MMM<sup>+</sup>15] and references therein.

Vafa mirror here. However, the M-theory uplift of the Katz–Klemm–Vafa geometric engineering to five compactified dimensions [KKV97, LN98] suggests that the planar part of the topological string free energy should be governed by special geometry on a family of  $5d$  Seiberg–Witten curves (Section 3.5), with gauge group  $\mathcal{G}_\Gamma$  specified by the ADE label of  $\Gamma$  via the McKay correspondence. Furthermore, in light of the connection of  $4d$  pure  $\mathcal{N} = 2$  Yang–Mills theory with the classical ADE Toda chain, it is natural to speculate that the  $5d$  curves should arise as the spectral curves of some relativistic deformation of the Toda chain, as has been known for a long time for the case  $\mathcal{G} = \mathrm{SU}(p)$  [Rui90, Nek98]. We will then be compelled to propose that the B-model target space will be given by the family of spectral curves of the Toda-type classical integrable system recently constructed in [Wil13, FM14] on the double Bruhat cells of the loop group  $\hat{\mathcal{G}}$ , as we recall in Section 3.6. Concretely, the Toda spectral curves take the form

$$\mathcal{P}_{\mathcal{G}^\#}^{\mathrm{Toda}}(X, Y; u) = \det [Y \mathbf{1} - \rho_{\min}(L_{\mathbf{w}}^{\mathcal{G}^\#})] = 0, \quad (1.1)$$

where  $L_{\mathbf{w}}^{\mathcal{G}^\#}$  is the Lax matrix of the Toda system on a suitable cell  $\mathbf{w}$  of the affine co-extended group  $\mathcal{G}^\#$ ,  $\rho_{\min}$  is an irreducible representation of  $\mathcal{G}$  of minimal dimension, and  $X \in \mathbb{C}^*$  is the spectral parameter of the Lax matrix. The right-hand side expands in the spectral invariants of the Lax matrix, which are encoded in  $R = \mathrm{rank}(\mathcal{G})$  independent parameters  $u = (u_1, \dots, u_R)$  – these are the Hamiltonians for the Toda classical integrable system on  $\mathcal{G}$ , and they correspond to the classical Weyl-invariant order parameters of the gauge theory vacua. We also have one additional parameter  $u_0$  associated to the affine root of  $\mathcal{G}^\#$ , which plays the role of the speed of light in the mechanical system, and is related to the exponentiated volume of the base  $\mathbb{P}^1$  in the mirror A-model.

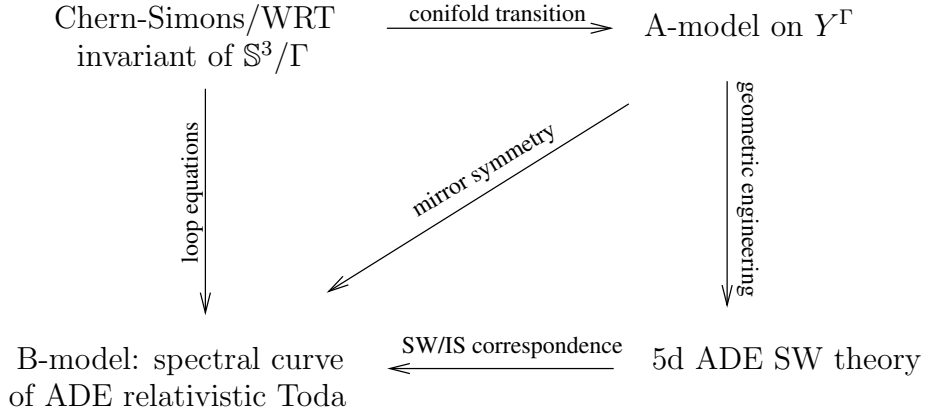


Figure 1: The chain of dualities behind our proposal.

On the other hand, the  $\mathfrak{sl}_{N+1}$  evaluation of the LMO invariant of the Seifert space  $\mathbb{S}^\Gamma = \mathbb{S}^3/\Gamma$  is given by the partition function of a random matrix model, and observables in this matrix model encode perturbative quantum invariants of fiber knots; to disambiguate notations,  $\mathcal{D}_\Gamma$  in the following denotes a Dynkin diagram of type  $A$ ,  $D$  or  $E$ , bijectively associated with the cyclic or binary polyhedral group  $\Gamma$  (A-model), and also with the compact, simply connected, simply-laced Lie group  $\mathcal{G}_\Gamma$  (B-model). This matrix model has a spectral curve:

$$\mathcal{P}_{\mathcal{D}_\Gamma}^{\mathrm{LMO}}(X, Y; \hat{\lambda}) = 0, \quad \lambda \triangleq \hat{\lambda}/\sigma = N\hbar/\sigma, \quad (1.2)$$

which depends on the three-dimensional geometry only via  $\Gamma$  and the Seifert invariant  $\sigma$  defined in (2.1). The latter only appears in the definition of the renormalized 't Hooft parameter  $\lambda$ . The all-order asymptotic expansion of the partition function and observables can be obtained from the topological recursion of Eynard–Orantin [EO07] applied to this curve – this is a B-model computation, in view of the remodeling proposal [BKMP09].

We propose that, for a suitable restriction  $u = u(\lambda)$  of parameters on the Toda side, the curves  $\mathcal{P}_{\mathcal{D}_\Gamma}^{\text{LMO}}$  and  $\mathcal{P}_{\mathcal{G}_\Gamma}^{\text{Toda}}$  agree up to an abelian factor  $(Y - 1)^\bullet$ . Therefore, the generating functions of LMO invariants and perturbative quantum invariants of fiber knots in  $\mathbb{S}^\Gamma$  receive an interpretation as suitably restricted Gromov–Witten/Donaldson–Thomas partition functions of  $Y^\Gamma$ . Our proposal passes several non-trivial tests, and automatically retrieves the known results for the case of lens spaces  $L(1, p)$ , where  $\Gamma = \mathbb{Z}/p\mathbb{Z}$  is a cyclic group,  $Y^\Gamma$  is a toric variety, and  $\mathcal{G} = \text{SU}(p)$ . The non-toric cases have so far remained unexplored, and they are the main focus of this paper.

## 1.2 Summary of results and organization of the article

We now describe more precisely our results and their mathematical status. They can be grouped in three main strands.

Firstly, we construct the A-model geometries  $Y^\Gamma$  by a direct generalization of the geometric transition for the case of spherical Seifert spaces (Section 3.1). We also highlight an extension of the holomorphic disk counting of [AV00, KL02, BC11] to the non-abelian orbifold case, and introduce the generating functions of open/closed Gromov–Witten invariants that are relevant for the discussion. Secondly, we propose (Section 3.5) and carry out the detailed construction of the spectral curves (1.1) for  $\mathcal{G} = A, D, E_6, E_7$ ; this requires substantial work and occupies the bulk of Section 6. For  $\mathcal{G} = E_8$ , computational complexity restricts the amount of data we can extract, while still allowing us to make some universal predictions on the form of (1.1), as well as a complete derivation of the spectral curve at the special point in moduli space corresponding to the  $\Gamma$ -orbifold of the conifold. Thirdly, for all  $\mathcal{G} \neq E_8$ , combining these results with [BE14], we can establish that the  $\mathfrak{sl}_{N+1}$  LMO invariants of  $\mathbb{S}^\Gamma$  are computed by the Eynard–Orantin invariants of the Toda curves, which may be regarded as a restricted, B-model version of the GOV correspondence; for  $\mathcal{G} = E_8$ , a complete proof is out of reach of our methods, but we propose it as a conjecture passing non-trivial checks.

The strategy we employ in our proof runs as follows: the LMO invariant on any Seifert space has been computed in [BNL04, Mar04], and for weight system  $\mathfrak{sl}_{N+1}$  it takes the form of the partition function for a random  $N \times N$  hermitian matrix model. The authors of [BE14] rely on [BGK15] to prove the existence of an asymptotic expansion when  $N \rightarrow \infty$ , and on [BEO15] to show that the latter is computed by the topological recursion applied to the spectral curve  $\mathcal{P}_{\mathcal{D}_\Gamma}^{\text{LMO}}$  of the matrix model. This material is reviewed in Section 2.4. The computation of the LMO spectral curve occupies Section 5, and is completed for  $A, D$  and  $E_6$ , while the result for  $E_7$  and  $E_8$  involves a number of parameters, in principle fixed by algebraic constraints that we could not solve. We however point out that, compared to [BE14], the complete expression of the LMO curve for  $E_6$  is new (Section 5.5).

Our comparison statement is presented in Section 4 and implemented in Section 6. It boils down to a general recipe to identify the function  $u(\lambda)$  such that the LMO spectral curve and the Toda spectral curve specialized to  $u \leftarrow u(\lambda)$  coincide (Conjecture 4.1). We give the expression of  $u(\lambda)$  for  $\mathcal{D} \in \{A, D, E_6\}$ , thus proving the conjecture. Algebraic complexity challenges the computation for  $E_7$  and  $E_8$ , but we are however able to prove that the specialization exists for  $E_7$ , and we find exact agreement of the Toda/LMO curves at the conifold point (i.e.  $\lambda = 0$  on the LMO side) for  $E_8$ , as well as a more general equality of their vertical slope polynomial. This comparison pertains to the left vertical arrow in Figure 1, and can be formulated as follows (see Section 2 for the relevant notation):

**Proposition 1.1** *Let  $(E_j[X; u])$  be the eigenvalues of the Lax matrix of the Toda integrable system for the affine co-extended group of type ADE, specified by fundamental character values  $u_1, \dots, u_R$ , Casimir  $u_0 = -\exp(-\chi_{\text{orb}}\lambda/2)$  and spectral parameter  $X$ . There exist a specialization  $(u_i(\lambda))_i$  and an explicit vector  $\hat{v}_j \in \mathbb{Z}^a$  such that the Taylor expansion of  $E_j[X; u(\lambda)]$  near  $X \rightarrow \infty$  is equal to:*

$$\mathcal{Y}_{\hat{v}_j}(X) = \prod_{\ell=0}^{a-1} [\mathcal{Y}(e^{2i\pi\ell/a} X^{1/a})]^{\hat{v}_j(\ell)}, \quad (1.3)$$

with:

$$\mathcal{Y}(X) = -X^{1/a} c \exp\left(\frac{\chi_{\text{orb}}\lambda}{a} \sum_{k \geq 0} X^{-k/a} \langle \text{Tr } \mathcal{U}^k \rangle^{(0)}\right), \quad (1.4)$$

and where  $\langle \text{Tr } \mathcal{U}^k \rangle^{(0)}$  is the large  $N$  limit of the moments of the random matrix  $\mathcal{U}$  in the Seifert matrix model. Furthermore, the full  $1/N$  asymptotic expansion of  $\langle \text{Tr } \mathcal{U}^{k_1} \dots \text{Tr } \mathcal{U}^{k_n} \rangle_{\text{conn.}}$  is computed from (1.3)–(1.4) by the Eynard–Orantin recursion [EO07]. For  $k_i \in (a/a_m)\mathbb{Z}$ , this is identified with the  $1/N$  expansion of the perturbative quantum invariants (in virtual  $k$ -th power sum representation) of the knot going along the fiber of order  $a_m$  in the Seifert manifold.

**Remark 1.1** Combining the results of [Han01] and [Mar04], one sees that the matrix model observables described in Section 2.4 appear as one term in the expression of the  $\mathfrak{sl}_{N+1}$  quantum invariants of fiber knots in Seifert manifolds produced by the Witten–Reshetikhin–Turaev–Wenzl TQFT at roots of unity; in Chern–Simons theory, localization heuristics identifies it with the contribution of the trivial flat connection to the Chern–Simons path integral. Throughout the paper, we will use the name “perturbative quantum invariants” to refer to these quantities. Whenever the trivial connection is isolated, i.e. for lens spaces and the Poincaré sphere, these should coincide with the dominant contribution to the saddle-point asymptotics of Wilson loops in Chern–Simons theory.

For the  $A$ -series, this correspondence has been known to extend to the perturbative expansion in Chern–Simons theory around a general flat connection [HY09]. Its formal analogy with the general simply-laced case cries out for generalization to the  $D$ - and  $E$ -series, and we speculate in Conjecture 4.2 on extending our statements to an arbitrary flat background.

The link between the  $A$ - and the  $B$ - model geometry – i.e. the diagonal arrow in Figure 1 – will be explored in a subsequent publication [Bri15], where more details can be found on the computations leading to the results of Section 6.

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## 2 Chern-Simons theory and Seifert spaces

This section reviews the main characters in our play, starting from the LMO invariants and Chern–Simons theory of Seifert 3-manifolds (Section 2), and in particular the spherical ones. We also discuss rigorous aspects of the matrix model approach. Then, we argue on physical grounds using large  $N$  dualities, geometric transitions (Section 3.1) and geometric engineering (Section 3.5), how Chern–Simons theory on  $\mathbb{S}^{\text{ADE}}$  relates to  $d = 5$ ,  $\mathcal{N} = 1$  pure Yang–Mills theory with ADE gauge group, and in turn to the classical integrable systems that govern its effective action up to two derivatives (Section 3.6). This is the necessary material to present our two main conjectures in Section 4.

### 2.1 Geometry of Seifert 3-manifolds

Seifert fibered spaces are manifolds  $M^3$  that are  $\mathbb{S}^1$ -bundles over orbifold surfaces [Sei80]. When the base surface is the sphere  $\mathbb{S}^2$  with  $r$  orbifold points of order  $a_1, \dots, a_r$ ,  $M^3$  can be realized by rational surgery on the link in  $\mathbb{S}^3$ , consisting of one main component passing through  $r$  meridians. The surgery slopes are  $1/b$  on the main component, and  $a_m/b_m$  on the  $m$ -th meridian. Here,  $a_m > 0$  and  $0 \leq b_m < a_m$  is coprime to  $a_m$ . There exist moves changing the surgery data but giving isomorphic Seifert spaces. Nevertheless, the tuple  $(a_1, \dots, a_r)$  and

$$\sigma \triangleq b + \sum_{m=1}^r \frac{b_m}{a_m} \quad (2.1)$$

are invariants of Seifert fibered spaces. For  $r \geq 3$ ,  $(a_1, \dots, a_r)$  is a topological invariant of  $M^3$ , whereas the cases  $r = 1$  or  $2$  realize lens spaces in several inequivalent ways as Seifert fibered spaces. Two quantities are particularly important:

$$a \triangleq \text{lcm}(a_1, \dots, a_r), \quad \chi_{\text{orb}} \triangleq 2 - r + \sum_{m=1}^r \frac{1}{a_m}. \quad (2.2)$$

A presentation of the fundamental groups of Seifert spaces was described in [Sei80] and the fundamental groups identified in [Orl72]: we remind this in Appendix A. The key fact is that



$\pi_1(M^3)$  is finite iff  $\chi_{\text{orb}} > 0$  and  $\sigma \neq 0$ ; this occurs for lens spaces or for  $r = 3$  exceptional fibers of order  $(2, 2, p)$ ,  $(2, 3, 3)$ ,  $(2, 3, 4)$ ,  $(2, 3, 5)$ . Then, the orbifold fundamental groups of the  $2d$ -base of the Seifert fibration is the spherical triangle group  $\Gamma = (a_1, a_2, a_3)$ . The resulting 3-manifolds  $\mathbb{S}^\Gamma \triangleq \Gamma \backslash \mathbb{S}^3$  are *spherical Seifert spaces*: these are quotients of the 3-sphere by a finite group of isometries acting smoothly, linearly and freely. Up to central extension, as reviewed in Appendix A-B, the list of possible groups is exhausted by  $\Gamma \subset \text{SL}(2, \mathbb{C})$  being a cyclic or binary polyhedral group. By the McKay correspondence [McK80], these have an ADE classification given in Table 1. Throughout the text, we will employ the labeling by ADE Dynkin diagrams  $\mathcal{D}_\Gamma$  to refer to the corresponding Seifert geometry.

Exceptional fibers	$\Gamma$	$\mathcal{D}_\Gamma$
$(p)$	$\mathbb{Z}/p\mathbb{Z}$	$A_{p-1}$
$(2, 2, p)$	$\text{Q}_{4(p+2)}$	$D_{p+2}$
$(2, 3, 3)$	$\text{P}_{24}$	$E_6$
$(2, 3, 4)$	$\text{P}_{48}$	$E_7$
$(2, 3, 5)$	$\text{P}_{120}$	$E_8$

Table 1: ADE labeling of spherical Seifert manifolds.  $\text{Q}_{4p}$  is the binary dihedral group, of order  $4p$ ;  $\text{P}_{24}$ ,  $\text{P}_{48}$  and  $\text{P}_{120}$  denote the binary tetra-, octa-, and icosahedral groups respectively.

As  $\pi_1(\mathbb{S}^\Gamma) = \Gamma$  is finite,  $H_1(\mathbb{S}^\Gamma, \mathbb{Z})$  is purely torsion and  $\mathbb{S}^\Gamma$  is always a rational homology sphere (QHS). In our list, the only case where we obtain an integer homology sphere is the  $E_8$  case with  $b_1 = b_2 = b_3 = -b = 1$ : this is the Poincaré sphere.

## 2.2 LMO invariant

Before getting to Chern–Simons theory in Section 2.4, we first present the mathematical avatar about which this article is mainly concerned: the LMO invariant [LMO98]. It is a graph-valued formal series associated to any rational homology sphere. The choice of a simple Lie algebra  $\mathfrak{g}$  gives an evaluation of the graphs, and converts this series into a formal series with rational coefficients:

$$\ln \mathcal{Z}_{\text{LMO}}(M^3) = \sum_{g \in \mathbb{N}/2} \hbar^{2g-2} \mathcal{F}_g(M^3) \in \hbar^{-2} \mathbb{Q}[[\hbar]]. \quad (2.3)$$

Bar-Natan and Lawrence [BNL04] obtained a surgery formula allowing them to compute the LMO invariant of Seifert manifolds which are QHS, and after picking up a simple Lie algebra, the result takes the form:

$$\mathcal{Z}_{\text{LMO}}^{\mathfrak{g}}(M^3) = C_h^{\mathfrak{g}}(M^3) \int_{\mathfrak{h}} d\phi \prod_{\alpha > 0} (\sinh[(\alpha \cdot \phi)/2])^{2-r} \prod_{m=1}^r \sinh[(\alpha \cdot \phi)/2a_m] e^{-\phi^2/(2\sigma\hbar)}. \quad (2.4)$$

$\mathfrak{h}$  is the (real) Cartan subalgebra of  $\mathfrak{g}$ , the product ranges over all positive roots, and  $(x, y) \mapsto x \cdot y$  is the Killing bilinear form, and  $d\phi$  the corresponding Riemannian volume.  $C_h^{\mathfrak{g}}(M^3)$  is an explicit prefactor involving  $a_m$ ,  $\sigma$  and the Casson–Walker invariant of  $M^3$  [Wal92]. Apart from this



contribution, the only dependence on  $b_m$  is hidden in the parameter  $\sigma$  defined in (2.1).

We will be mainly interested in the weight system of the Lie algebra  $\mathfrak{sl}_{N+1}$ . In this case, elementary combinatorics shows that the LMO invariant can be repackaged by setting  $\hat{\lambda} = N\hbar$  into a well-defined formal series:

$$\ln \mathcal{Z}_{\text{LMO}}^{\mathfrak{sl}(N+1)}(M^3) = \sum_{g \in \mathbb{N}} N^{2g-2} \mathcal{F}_g(\hat{\lambda}; M^3), \quad \mathcal{F}_g(M^3; \hat{\lambda}) \in \mathbb{Q}[[\hat{\lambda}]]. \quad (2.5)$$

$\mathcal{F}_h$  are called the free energies. In the case of Seifert manifolds, we prefer to define:

$$\lambda \triangleq N\hbar/\sigma = \hat{\lambda}/\sigma, \quad (2.6)$$

and (2.4) for Seifert spaces becomes:

$$\mathcal{Z}_{\text{LMO}}^{\mathfrak{sl}_{N+1}}(M^3) = C_h^{\mathfrak{sl}_{N+1}}(M^3) \int_{\mathbb{R}^N} \prod_{1 \leq i < j \leq N} (\sinh[(\phi_i - \phi_j)/2])^{2-r} \prod_{m=1}^r \sinh[(\phi_i - \phi_j)/2a_m] \prod_{i=1}^N e^{-N\phi_i^2/2\lambda} d\phi_i. \quad (2.7)$$

### 2.3 The matrix model approach

The right-hand side of (2.7) provides a definition for a function of an integer  $N$  and a positive parameter  $\hat{\lambda}$ , that we denote  $Z_N(M^3; \hat{\lambda})$ . This is a convergent matrix integral, and its large  $N$  asymptotic behavior for a fixed  $\hat{\lambda} > 0$  can be studied rigorously with the techniques recently developed in [BGK15]. The main result of [BGK15] relies on an assumption of strict convexity, which is here satisfied when  $\chi_{\text{orb}} > 0$  and  $\hat{\lambda} > 0$  is small enough. One then obtains, for any  $g_0 \geq 0$ , an asymptotic expansion of the form:

$$\begin{aligned} Z_N(M^3; \lambda) &\triangleq \int_{\mathbb{R}^N} \prod_{1 \leq i < j \leq N} (\sinh[(\phi_i - \phi_j)/2])^{2-r} \prod_{m=1}^r \sinh[(\phi_i - \phi_j)/2a_m] \prod_{i=1}^N e^{-N\phi_i^2/2\lambda} d\phi_i \\ &= N^{N+5/12} \exp \left( \sum_{g=0}^{g_0} N^{2-2g} F_g(\lambda; M^3) + \mathcal{O}(N^{2-2g_0}) \right), \end{aligned} \quad (2.8)$$

and  $F_g(M^3; \lambda)$  extends as an analytic function of  $\lambda$  in a vicinity of 0. It was proved in [BEO15] that the  $F_g$  are computed by the topological recursion of [EO07]. This requires only the knowledge of the spectral curve of the matrix model, here conveniently defined as:

$$W_{0,1}(x) \triangleq \lim_{N \rightarrow \infty} \frac{1}{N} \left\langle \sum_{i=1}^N \frac{x}{x - e^{\phi_i/a}} \right\rangle, \quad (2.9)$$

and the knowledge of the two-point function:

$$W_{0,2}(x_1, x_2) \triangleq \lim_{N \rightarrow \infty} \left\{ \left\langle \sum_{i_1, i_2=1}^N \frac{x_1 x_2}{(x_1 - e^{\phi_{i_1}/a})(x_2 - e^{\phi_{i_2}/a})} \right\rangle - \left\langle \sum_{i_1=1}^N \frac{x_1}{x_1 - e^{\phi_{i_1}/a}} \right\rangle \left\langle \sum_{i_2=1}^N \frac{x_2}{x_2 - e^{\phi_{i_2}/a}} \right\rangle \right\}. \quad (2.10)$$

It turns out that  $W_{0,2}(x_1, x_2)$  can be analytically continued as a meromorphic function of 2 variables in the same curve, i.e. on  $\{(x_1, y_1, x_2, y_2) \in \mathbb{C}^4, \ y_i = W_{0,1}(x_i)\}$ . The topological

recursion then provides a universal algorithm to compute the whole  $1/N$  asymptotic expansion of correlation functions, and then  $F_g$  for  $g \geq 2$ . Beyond computations which are anyway rather heavy to perform explicitly, we learn that, to understand the singularities of the continuation of  $(F_g)_{g \geq 2}$ ,  $\partial_\lambda F_1$  and  $\partial_\lambda^2 F_0$  as an analytic function of  $\lambda$  in the complex plane, it is enough to understand the singularities of the analytic family of curves  $\{y = W_{0,1}(x)\}_\lambda$ .

**Remark 2.1** One may ask what these analytic functions  $F_g(\lambda)$  in (2.8) have to do with the formal series  $\mathcal{F}_g(\lambda)$  in (2.5). It can be proved that the Taylor series of  $F_g(\lambda)$  at  $\lambda \rightarrow 0$  gives  $\mathcal{F}_g(M^3; \lambda)$ . Indeed, it is easy to show that the formal series  $\mathcal{F}_g(M^3; \lambda)$  satisfy some loop equations (let us call them formal), expressing them as generating series of a certain set of ribbon graphs with Boltzmann weights prescribed by (2.4), and these equations have a unique solution (see e.g. [Bor14]). It is also well-known that  $Z_N(M^3; \lambda)$  satisfies a set of loop equations, obtained for instance by integration by parts in the matrix model. Inserting the form of the asymptotic expansion (2.8) in these equations, collecting the powers of  $N$ , and collecting order by order in the Taylor expansion when  $\lambda \rightarrow 0$ , we obtain the same formal loop equations that were satisfied by  $\mathcal{F}_g(\lambda)$ . We can then conclude by uniqueness of the solution of the formal loop equations.

To recap, the matrix model and the study of  $F_g(\lambda)$  give a method to compute and establish convergence properties and analytic continuation of the formal series  $\mathcal{F}_g(\lambda)$ . The main task lies in the computation of the spectral curve, which was mainly addressed in [BE14] by one of the authors. It turns out that among Seifert spaces, only the ADE geometries have an algebraic spectral curve, with a subtlety that will be explained in Section 5.1. In Section 5.3 we review the construction of the matrix model spectral curves, which consists in describing the monodromy group of  $W(x)$ , and exhibiting the unique function that admits the singular behavior and branchcuts required by the problem.

## 2.4 Chern–Simons theory

In physics, the LMO invariant captures the  $\hbar \rightarrow 0$ , perturbative expansion of the Chern–Simons functional integral on  $M^3$  with compact, simply-connected gauge group  $G = \exp(\mathfrak{g})$ ,

$$Z_{\text{CS}}^{\mathfrak{g}}(k, M^3) = \int_{\mathcal{A}/\mathcal{G}} [\mathcal{D}\mathcal{A}] \exp\left(\frac{ik}{2\pi} \text{CS}[\mathcal{A}]\right), \quad (2.11)$$

$$\text{CS}[\mathcal{A}] = \int_{M^3} \left( \mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A}^3 \right), \quad (2.12)$$

around the trivial flat connection,  $\mathcal{A} = g dg^{-1}$ ; here  $k \in \mathbb{N}^*$  is the Chern–Simons level, and the LMO variables are identified as  $\hbar = 2i\pi/(k + h^\vee)$ ,  $\hat{\lambda} = h^\vee \cdot \hbar$  with  $h^\vee$  the dual Coxeter number of  $\mathfrak{g}$ . The full partition function  $Z_{\text{CS}}^{\mathfrak{g}}$  of Chern–Simons of Seifert manifolds that are QHS can be found in various ways, depending on the mathematical starting point one chooses for Chern–Simons theory – which morally realize the path integral with Chern–Simons action. They all lead to the same answer for Seifert spaces, and  $Z_{\text{LMO}}^{\mathfrak{g}}$  appears as one term within  $Z_{\text{CS}}^{\mathfrak{g}}$ .

In a Hamiltonian context, Mariño [Mar04] cleverly used the gluing rules of the Wess–Zumino–Witten TQFT, the Kac–Peterson formula for the  $S$ - and  $T$ -matrices, and the surgery presentation of Seifert spaces to derive the formula (2.4) for  $Z_{\text{LMO}}^{\mathfrak{g}}$ . His work generalized to all simply-laced Lie algebra an observation of Lawrence and Rozansky [LR99] for  $\mathfrak{sl}_2$ , and can be seen as the TQFT

analogue of the surgery approach of [BNL04]. His matrix model representation has then been rederived via functional localization, either by exploiting the  $\mathbb{S}^1$ -action of the Seifert fibration to reduce (2.11) to a discrete sum over flat connections over the orbifold sphere [BT06, BT13], or by taking a choice of a contact structure on  $M^3$  and resorting to non-abelian localization [BW05, Bea13] to single out the contribution of isolated flat connections<sup>4</sup>, or yet again [K11] by employing localization in a topological twist of a parent supersymmetric theory [KWY10]. The authors of [Bea13, K11, BT13] also show that the insertion of a Wilson line  $\mathcal{W}_{\mathcal{R}}(K_{a_m})$  along the exceptional fiber of order  $a_m$  decorated with a representation  $\mathcal{R}$  is represented in terms of  $\phi \in \mathfrak{h}$  as an insertion of the character  $\text{ch}_{\mathcal{R}}(e^{\phi_1/a_m}, \dots, e^{\phi_N/a_m})$ . For  $\mathfrak{g} = \mathfrak{sl}$ , the characters are the symmetric functions, and the definition of  $\mathcal{W}_{\mathcal{R}}$  can be extended by linearity to the whole character ring. If we restrict to the contribution of the trivial flat connection, a good way to encode all of them at the same time is to define the correlators of the matrix model. The latter are defined, for  $n \geq 1$ , as:

$$W_n(x_1, \dots, x_n) \triangleq \left\langle \prod_{j=1}^n \sum_{i_j=1}^N \frac{x}{x - e^{\phi_{i_j}/a}} \right\rangle_{\text{conn.}} \quad (2.13)$$

with respect to the measure in (2.8), and they depend implicitly on  $\lambda$ . For our purposes, it is helpful to work with connected observables, as they enjoy a well-defined  $1/N$  expansion. For  $k$  an integer, let  $p_k$  be the  $k$ -th power sum character. Then, we have:

$$\left\langle \prod_{j=1}^n \mathcal{W}_{p_{k_j}}(K_{a_{m_j}}) \right\rangle_{\text{conn.}} = \left[ \prod_{j=1}^n x^{-k_j(a/a_{m_j})} \right] W_n(x_1, \dots, x_n). \quad (2.14)$$

We can read off invariants of knots going along the various exceptional fibers  $K_{a_m}$  by looking at the coefficients of expansion of the correlators when  $x_i \rightarrow \infty$  (or  $x_i \rightarrow 0$ ) for orders that are multiples of  $a/a_m$ .

The discussion of Section 2.2 applies to the  $W_n$  as well. For the spherical Seifert geometries, the work of [BGK15] establishes an asymptotic expansion when  $N \rightarrow \infty$ :

$$W_n(x_1, \dots, x_n) = \sum_{g \geq 0} N^{2-2g-n} W_{g,n}(x_1, \dots, x_n) \quad (2.15)$$

at least for  $\lambda > 0$  small enough. The coefficient of  $x^{-k(a/a_m)}$  in the Laurent expansion at infinity of the function:

$$W(x) \triangleq W_{0,1}(x) \quad (2.16)$$

defining the spectral curve computes the planar limit of the HOMFLY invariant of  $K_{a_m}$  colored with the virtual character  $p_k$ . The other coefficients do not seem to have an interpretation in terms of 3d topology, but they do influence the monodromy of the spectral curve<sup>5</sup>.

<sup>4</sup>It should be stressed that the trivial flat connection is isolated only in the case of lens spaces and the  $E_8$  Seifert geometry. Therefore, identifying  $Z_{\text{LMO}}^{\mathfrak{g}}$  with the trivial connection is only legitimate in those cases.

<sup>5</sup>In the case of lens spaces, invariants of fiber knots are related to invariants of torus knots in  $\mathbb{S}^3$ . We point out that [JKS14] defines and compute a new spectral curve that only contains the physical part of the information (i.e. the planar limit of HOMFLY's of the torus knots) skipping the other coefficients. They are able to find a (very complicated) 2-point function which, after applying topological recursion, still gives the “physical part” of the correct higher genus expansion. From a conceptual point of view, it is simpler to keep on with spectral curves that

### 3 Construction of $Y^\Gamma$ and topological string dualities

#### 3.1 Topological large $N$ duality

As for any quantum gauge theory with gauge group  $U(N)$  and fields in the adjoint representation, the formal perturbative expansion of the Chern–Simons path integral can be formulated as an expansion in ribbon graphs  $G$ , whose dual graphs are triangulations of a closed oriented topological 2-manifold  $S_G$ . Elementary combinatorics then shows that each loop in the diagram contributes a factor of  $\hat{\lambda} = g_{\text{YM}}^2 N$ , and the overall topology contributes a factor of  $g_{\text{YM}}^{-2\chi(C_G)}$  [tH74]. In particular, the perturbative free energy takes the form

$$\mathcal{F}_{\text{CS}}^{\text{fl}_{N+1}}(M^3; g_{\text{YM}}) = \sum_{g,n \geq 0} \mathcal{F}_{g,n}(M^3) \hat{\lambda}^n g_{\text{YM}}^{4g-4} \in g_{\text{YM}}^{-4} \mathbb{Q}[[\hat{\lambda}, g_{\text{YM}}^4]]. \quad (3.1)$$

For the case of  $U(N)$  Chern–Simons theory on a closed oriented 3-manifold  $M^3$ , Witten showed [Wit95] that this can be reinterpreted as the target string field theory of the open topological A-model on the cotangent bundle  $T^*M^3$ , with  $N$  Lagrangian A-branes wrapping the image of the zero section (see [Mar05] for a review). Here, the string coupling constant should be identified with  $g_s = g_{\text{YM}}^2$ ; in particular, the ribbon graph expansion translates into a virtual count of open holomorphic worldsheet instantons with A-type Dirichlet boundary condition on  $M^3$ . A formal resummation of the contribution of the connected contribution of the boundary – the “holes” in the worldsheet – gives rise to a formal closed string expansion,

$$\mathcal{F}_{\text{CS}}^{\text{fl}_{N+1}}(M^3, g_{\text{YM}}) = \sum_{g \geq 0} g_s^{2g-2} \cdot \hat{\lambda}^{-(2g-2)} \mathcal{F}_g(M^3; \hat{\lambda}), \quad (3.2)$$

When  $M^3 = \mathbb{S}^3$ , Gopakumar and Vafa identified the closed string model as the closed topological A-model on the resolved conifold  $\text{Tot}[\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1]$ : here  $g_s$  is the closed string coupling constant, and  $\hat{\lambda}$  is identified with the Kähler parameter of the base  $\mathbb{P}^1$ . Geometrically, this target space is obtained from  $T^*\mathbb{S}^3$  by a complex degeneration to a normal singular variety (the singular conifold) obtained by contracting the base  $\mathbb{S}^3$ , followed by a minimal crepant resolution of the resulting singularity with a  $\mathbb{P}^1$  as its exceptional locus. While there are obstructions to extend this circle of ideas to more general 3-manifolds [BGST10], it is still natural to conjecture, in view of the positive results of [HY09], that the same scenario could apply to the case of spherical Seifert manifolds and  $\Gamma \subset \text{SU}(2)$  quotients of the conifold, as we now describe.

#### 3.2 Geometric transition for $\mathbb{S}^3$

Let us review the conifold transition for the simplest case of  $\mathbb{S}^3$  with unit radius. Since  $\mathbb{S}^3 \simeq \text{SU}(2)$  is a Lie group,  $T^*\mathbb{S}^3$  is a trivial bundle; its fiber at identity is the space  $i\mathcal{H}_0(2, \mathbb{C})$  of traceless anti-hermitian  $2 \times 2$  matrices. Any matrix  $A \in \text{GL}(2, \mathbb{C})$  can be written uniquely by polar decomposition  $M = Ue^H$  where  $U \in \text{U}(2)$  and  $H \in \mathcal{H}(2, \mathbb{C})$  definite positive, and if we restrict to  $\det(A) = 1$ , we must have  $\det(U) = 1$  and  $\text{tr}(H) = 0$ . Therefore, the polar decomposition gives an isomorphism:

$$T^*\mathbb{S}^3 \stackrel{\ell}{\simeq} \text{SL}(2, \mathbb{C}) \quad (3.3)$$

---

may contain knot-theoretic irrelevant information, which are used to get the higher genus corrections, and only then discard coefficients which do not have a knot-theoretic interpretation. The equivalence between the two approaches is guaranteed by a property of commutation with “forgetting information” enjoyed by the topological recursion, see [BEO15].

This description can be fit into a flat family  $\psi : X = \mathrm{GL}(2, \mathbb{C}) \rightarrow \mathbb{C}^*$  given by the determinant map. Then the fiber  $X_{[\mu]}$  at a point  $\mu$  such that  $\mathrm{Im} \mu = 0$  and  $\mathrm{Re} \mu > 0$  is isomorphic to the cotangent bundle  $T^*\mathbb{S}_{[\mu]}^3$  of a sphere with radius  $\mu$ . Explicitly, writing

$$\rho(A) = w_0 + i\vec{w} \cdot \vec{\sigma}, \quad w_j = p_j + iq_j \quad (3.4)$$

realizes  $X_{[\mu]}$  as the real complete intersection in  $T^*\mathbb{R}^4$  cut out by  $\sum_{j=1}^4 q_j^2 - p_j^2 = \mu$ ,  $\sum_{j=1}^4 q_j p_j = 0$ .

Let us add the locus of non-invertible matrices to form:

$$\tilde{\psi} : \mathrm{Mat}(2, \mathbb{C}) \longrightarrow \mathbb{C}. \quad (3.5)$$

The fiber  $X_{[0]}$  above  $\mu = 0$  is the singular quadric  $\det A = 0$ . It admits a canonical minimal resolution

$$\pi : \widehat{X} \longrightarrow X_{[0]}, \quad \widehat{X} \triangleq \{(\rho(A), v) \in X_{[0]} \times \mathbb{P}^1, \quad \rho(A)v = 0\}, \quad (3.6)$$

where  $\pi$  is the projection to the first factor. The point  $A = 0$  is singular in  $X_{[0]}$ , and its fiber is a complex projective line with  $[v_1 : v_2]$  as homogeneous coordinates. Using coordinate charts on  $\mathbb{P}^1$  exhibits  $\widehat{X}$  as the total space of  $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \mathbb{P}^1$ , i.e. the resolved conifold. As a symplectic manifold, it supports a one-dimensional family of complexified Kähler forms coming from its presentation in (3.6), namely

$$\omega_{t_B} = i_1^* \omega_{\mathbb{C}^4} + t_B i_2^* \omega_{\mathrm{FS}}, \quad (3.7)$$

where  $i = (i_1, i_2)$  is the factorized form of the embedding  $i : X \hookrightarrow \mathrm{Mat}(2, \mathbb{C}) \times \mathbb{P}^1$  from (3.6), and  $\omega_{\mathbb{C}^4}$  and  $\omega_{\mathrm{FS}}$  are respectively the canonical Kähler form on  $\mathrm{Mat}(2, \mathbb{C}) \simeq \mathbb{C}^4 \simeq T^*\mathbb{R}^4$  and the Fubini–Study form on  $\mathbb{P}^1$ .

### 3.3 Geometric transition for $\mathbb{S}^\Gamma$

We now consider the action of finite groups of isometries of  $\mathbb{S}^3$ , reviewed in Appendix B. The morphism  $\rho$  is compatible with the isometric action of left and right multiplication on  $\mathbb{S}^3 \simeq \mathrm{SU}(2)$ . This means that, if we denote  $\tilde{\Phi}_4$  the lift of this action to an action by symplectomorphisms on  $T^*\mathbb{S}^3$ , we have for any  $(q_1, q_2) \in \mathrm{SU}(2) \times \mathrm{SU}(2)$  and any  $A \in T^*\mathbb{S}^3$ ,

$$\rho(\tilde{\Phi}_4(q_1, q_2) \cdot A) = q_1 \rho(A) q_2^{-1}. \quad (3.8)$$

Let us focus on the left action by a finite subgroup  $\Gamma \subset \mathrm{SU}(2)$ . This is a fiberwise action on  $\psi : X \rightarrow \mathbb{C}^*$ , which is free on each fiber  $X_{[\mu]}$ . When  $\mu > 0$ , we claim that the set of equivalence classes is just isomorphic to  $T^*\mathbb{S}^\Gamma$ . Indeed, consider the local diffeomorphism on  $\mathbb{R}^8$  given by

$$\begin{aligned} \tilde{p}_1 &= q_1 p_1 + q_2 p_2 + q_3 p_3 + q_4 p_4, & \tilde{p}_2 &= q_1 p_2 - q_2 p_1 + q_4 p_3 - q_3 p_4, \\ \tilde{p}_3 &= q_3 p_1 + q_4 p_2 - q_1 p_3 - q_2 p_4, & \tilde{p}_4 &= q_3 p_2 - q_4 p_1 - q_2 p_3 + q_1 p_4, \end{aligned} \quad (3.9)$$

and

$$\tilde{q}_i = \frac{q_i}{\sqrt{\mu + \sum_{j=1}^4 p_j^2}}. \quad (3.10)$$

It is non-singular everywhere for  $\mu > 0$ , and the resulting real sixfold is just  $\mathbb{R}^3 \times \mathbb{S}^3$ , cut out in  $\mathbb{R}^8$  by:

$$\tilde{p}_1 = 0, \quad \sum_{i=1}^4 \tilde{q}_i^2 = 1. \quad (3.11)$$

Using the generators of  $\Gamma$  given in Appendix B, it can be checked that the coordinates  $\tilde{p}_i$  are  $\Gamma$ -invariant so that the quotient is:

$$X_{[\mu]}^\Gamma = \frac{\text{Spec } \mathbb{C}[A]^\Gamma}{\langle \det A = \mu \rangle} \simeq \mathbb{R}^3 \times \mathbb{S}^\Gamma \quad (3.12)$$

which is isomorphic to  $T^*\mathbb{S}^\Gamma$  by Stiefel's theorem.

Now, let us look at the  $\Gamma$ -action on the resolution  $\hat{X}$ . It only acts on the first factor of (3.6), and hence this is a fiberwise action on  $p : \hat{X} \rightarrow \mathbb{P}^1$  (the second factor in (3.6)). The fiber at a point  $z \in \mathbb{P}^1$  is isomorphic to the du Val singularity  $\Gamma \backslash \mathbb{C}^2$ , and the resulting target geometry can be studied in two distinguished chambers of the stringy Kähler moduli space. Let

$$R \triangleq \text{rank}(\mathcal{G}_\Gamma). \quad (3.13)$$

In the orbifold chamber, we are looking at the orbifold A-model on  $Y_{\text{orb}}^\Gamma \triangleq [\Gamma \backslash \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}(-1)]$ . Its degree two orbifold quantum cohomology – i.e. the space of marginal deformations of the A-model chiral ring – is generated by classes  $(\delta, (\xi_j)_{j=1}^R)$ ; here  $\delta$  is the class of the base of  $[Y^\Gamma]_{[0]} \rightarrow \mathbb{P}^1$ , where  $[\cdot]_{[0]}$  denotes the untwisted sector, and  $\xi_j$  are twisted orbifold cohomology classes of Chen–Ruan degree two. In the large radius chamber, we take a crepant resolution  $Y_{\text{res}}^\Gamma$  of the singularities of  $Y_{\text{orb}}^\Gamma$  obtained by canonically resolving the surface singularity  $\Gamma \backslash \mathbb{C}^2$  fiberwise. The resulting Calabi–Yau threefold  $Y^\Gamma$  is thus an ALE fibration over  $\mathbb{P}^1$ , with fibers given by configurations of rational curves having normal bundle  $(0, -2)$ , and whose intersection matrix equates the negative of the Cartan matrix of  $\mathcal{G}_\Gamma$  [Rei]. Then  $H^2(Y_{\text{res}}^\Gamma)$  is generated as a vector space by the base class  $\delta$  above, plus classes  $(\gamma_j)_{j=1}^R$  representing the nodes in the chain of exceptional fiber  $\mathbb{P}^1$ 's. In the following we will often write  $Y^\Gamma$  to refer to either of the two chambers whenever the context applies to both of them.

### 3.4 A-model: Gromov–Witten theory on $Y^\Gamma$

In terms of the coordinates  $\{a_{ij} = \rho(A)_{ij}\}_{i,j=1,2}$  and  $[v_1 : v_2]$  of (3.4) and (3.6),  $Y^\Gamma$  supports a natural  $T \simeq \mathbb{C}^*$ -action given by

$$(a_{11}, a_{12}, a_{21}, a_{22}; [v_1 : v_2]) \longrightarrow (\mu a_{11}, a_{12}, \mu a_{21}, a_{22}; [\mu^{-1} v_1 : v_2]), \quad (3.14)$$

Here  $T$  acts trivially on the canonical bundle: on the full resolution  $Y_{\text{res}}^\Gamma$ , it has a compact fixed locus  $Y_{\text{res},T}^\Gamma$  consisting of two fibers above  $[0 : 1]$  and  $[1 : 0]$ , each isomorphic to a disjoint union of a  $\mathbb{P}^1$  with  $(R - 2)$  points; likewise, its fixed locus on  $Y_{\text{orb}}^\Gamma$  is the union of two  $\Gamma$ -orbifold points, i.e.  $Y_{\text{orb},T}^\Gamma \simeq B\Gamma \sqcup B\Gamma$ . The A-model/Gromov–Witten closed free energy of  $Y^\Gamma$  is then defined/computed

by localization [GP99]:

$$F^{\text{GW}}(Y^\Gamma) \triangleq \sum_{g \geq 0} g_s^{2g-2} F_g^{\text{GW}}(Y^\Gamma, t), \quad (3.15)$$

$$F_g^{\text{GW}}(Y^\Gamma, t) \triangleq \sum_{n=0}^{\infty} \sum_{\beta \in H^2(Y^\Gamma, \mathbb{Z})} \frac{\langle \Phi(t), \dots, \Phi(t) \rangle_{g, \beta}^{Y^\Gamma}}{n!}, \quad (3.16)$$

$$\langle \varphi_1, \dots, \varphi_n \rangle \triangleq \int_{[\mathcal{M}_{g,0}(Y_T^\Gamma, \beta)]^{\text{virt}}} \text{ev}_1^* \varphi_1 \cup \dots \cup \text{ev}_n^* \varphi_n \in \mathbb{Q}(\mu), \quad (3.17)$$

where  $\mu = c_1(\mathcal{O}_{BT}(1))$  denotes the equivariant parameter of  $T$  and  $\Phi$  is a cohomology class specified by linear coordinates  $t$  on  $H^\bullet(Y^\Gamma)$ . In fact, as the torus action is Calabi–Yau (i.e. it preserves the holomorphic volume form), Gromov–Witten invariants in positive degree (3.17) do not depend on  $\mu$  [MOOP08], nor do the higher genus invariants for  $g \geq 2$  and all  $\beta$ . Equations (3.15)–(3.17) will be our candidate for the A-model dual of the Chern–Simons free energy at large  $N$ .

## A-branes

The geometry of  $Y^\Gamma$  offers also a natural candidate for an A-model description of the large  $N$  expansion of the Wilson loops along fiber knots, (2.15), in terms of open Gromov–Witten invariants [KL02, BC11]. On the resolved conifold  $Y = Y^{\Gamma=\{1\}}$ , consider the anti-holomorphic involution  $\sigma : Y \rightarrow Y$  induced by  $\sigma(a_{22}) = \overline{a_{11}}$ ,  $\sigma(a_{21}) = \overline{a_{12}}$ . Equivalently, this means  $\sigma(w_{0,3}) = \overline{w_{0,3}}$ ,  $\sigma(w_{1,2}) = -\overline{w_{1,2}}$  in (3.4) and  $v_i \rightarrow \overline{v_{3-i}}$ . Its fixed locus is thus isomorphic to  $\mathbb{R}^2 \times \mathbb{S}^1$ , where the circle is given by the equator of the base  $\mathbb{P}^1$ , and it is Lagrangian with respect to the canonical Kähler form (3.7), as the first summand in (3.7) changes sign under  $\sigma$ , and the second vanishes as  $Y_\sigma \cap \mathbb{P}^1$  has non-vanishing codimension.

When  $\Gamma \subset \text{SU}(2)$  is cyclic, the  $\Gamma$ -action descends to a free action on the fixed locus  $Y_\sigma$ : this simultaneously defines Lagrangian branes on  $Y_{\text{orb}}^\Gamma$  and  $Y_{\text{res}}^\Gamma$  by respectively taking the orbit space  $Y_\sigma^\Gamma \triangleq \mathcal{L}_{\text{orb}}^\Gamma$  for  $Y_{\text{orb}}^\Gamma$ , and the transform  $\mathcal{L}_{\text{res}}^\Gamma$  of this condition under the resolution map for  $Y_{\text{res}}^\Gamma$ <sup>6</sup>. When  $\Gamma$  is non-abelian, on the other hand, the  $\Gamma$  action does not descend to an action on the  $\sigma$ -fixed locus, as can be checked directly on the generators (B.11)–(B.13). However,  $\Gamma$  preserves the symplectic form (3.7) on  $Y$  (see Appendix B), and one can check that the images of  $Y_\sigma$  under the degree 3 (resp. 2) generator  $j$  (resp.  $\kappa$ ) are Lagrangians having empty intersection with  $Y_\sigma$ . Then, defining

$$Y_\sigma^\Gamma = \begin{cases} Y_\sigma & \mathcal{D}_\Gamma = A, \\ Y_\sigma \sqcup \iota(Y_\sigma) & \mathcal{D}_\Gamma = D, \\ \bigsqcup_{\phi=\text{id}, j, j^2} \phi(Y_\sigma) & \mathcal{D}_\Gamma = E_6, E_7, \\ \bigsqcup_{\phi=\text{id}, j, j^2, \kappa} \phi(Y_\sigma) & \mathcal{D}_\Gamma = E_8, \end{cases} \quad (3.18)$$

the  $\Gamma$ -action descends on  $Y_\sigma^\Gamma$  to give Lagrangian branes  $\mathcal{L}_{\text{orb}}^\Gamma$  and  $\mathcal{L}_{\text{res}}^\Gamma$  as before. These branes have topology  $\mathbb{R}^2/(\mathbb{Z}/q_\Gamma \mathbb{Z}) \times \mathbb{S}^1$ , where  $q_\Gamma$  is tabulated in Table 2; notice that the  $\Gamma$ -action leaves the base  $\mathbb{P}^1$  unaffected (hence the  $\mathbb{S}^1$  factor) and that  $Y_\sigma^\Gamma$  is constructed from Lagrangian copies of  $Y_\sigma$  in the orbit of “non-cyclic” generators  $\iota$ ,  $j$  and  $\kappa$ , hence the  $\Gamma$ -action factors through a residual cyclic action on  $Y_\sigma$ , giving rise to a cyclic quotient of  $\mathbb{R}^2$ .

<sup>6</sup>We again omit the subscript from  $\mathcal{L}_{\text{res}}^\Gamma$  and  $\mathcal{L}_{\text{orb}}^\Gamma$  whenever the statements apply to both.



$\mathcal{D}_\Gamma$	$q_\Gamma$
$A_{p-1}$	$p$
$D_{p+2}$	$2p+4$
$E_{6,8}$	$4$
$E_7$	$8$

(3.19)

Table 2: Orders of the residual cyclic group action on  $Y_\sigma$  for  $\mathcal{D}_\Gamma = A_n, D_n, E_n$ .

As for the usual toric case, the Calabi–Yau torus action (3.14) allows then to define a virtual counting theory of stable open maps [BC11, KL02, Bri12] to  $Y^\Gamma$  having Dirichlet boundary conditions on  $\mathcal{L}^\Gamma$  via equivariant residues on  $\overline{\mathcal{M}}_{g,n}$  and  $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, \beta)$  (for  $Y_{\text{res}}^\Gamma$ ) or  $\overline{\mathcal{M}}_{g,n}(B\Gamma)$  (for  $Y_{\text{orb}}^\Gamma$ ):

$$\langle \varphi_1, \dots, \varphi_n \rangle_{g,n,\zeta,\vec{d}}^{Y^\Gamma, \mathcal{L}^\Gamma} \triangleq \int_{[\overline{\mathcal{M}}_{g,n}(Y^\Gamma, \mathcal{L}^\Gamma, \zeta, \vec{d})_T]^{\text{virt}}} \frac{\text{ev}_1^* \varphi_1 \cup \dots \cup \text{ev}_n^* \varphi_n}{e_T \left( N_{\overline{\mathcal{M}}_{g,n}(Y^\Gamma, \mathcal{L}^\Gamma, \zeta, \vec{d})_T}^{\text{virt}} \right)}. \quad (3.20)$$

Here,  $n$  is the number of connected components of the boundary of the source curve,  $\vec{d} = (d_1, \dots, d_n)$  with  $d_i \in H_1(\mathcal{L}^\Gamma)$  describe their winding around the equator, and  $\zeta \in H_2(Y^\Gamma, \mathcal{L}^\Gamma)$  is the relative homology class representing the image of the open worldsheet in  $Y^\Gamma$ . This can be packaged into formal generating series:

$$W_{g,n}^{\text{GW}}(Y^\Gamma, \mathcal{L}^\Gamma; t, w) \triangleq \sum_{n, \zeta, \vec{d}} \frac{\langle \Phi(t), \dots, \Phi(t) \rangle_{g,n,\zeta,\vec{d}}^{Y^\Gamma, \mathcal{L}^\Gamma}}{n!} \prod_{i=1}^n \frac{w_i^{d_i}}{d_i!}. \quad (3.21)$$

where  $t$  are again quantum cohomology parameters accounting for localized primary insertions. On the resolution, the divisor equation puts (3.16) and (3.21) in the form of the familiar worldsheet instanton expansion<sup>7</sup>

$$F_g^{\text{GW}}(Y_{\text{res}}^\Gamma, t) = \sum_{\beta \in H^2(Y_{\text{res}}^\Gamma, \mathbb{Z})} \langle 1 \rangle_{g,\beta}^{Y_{\text{res}}^\Gamma} e^{\beta \cdot t} \quad (3.22)$$

$$W_{g,n}^{\text{GW}}(Y_{\text{res}}^\Gamma, \mathcal{L}_{\text{res}}^\Gamma; t, w) = \sum_{\beta, \vec{d}} \langle 1 \rangle_{g,\beta,\vec{d}}^{Y_{\text{res}}^\Gamma, \mathcal{L}_{\text{res}}^\Gamma} e^{\beta \cdot t} \prod_{i=1}^n \frac{w_i^{d_i}}{d_i!}. \quad (3.23)$$

### 3.5 Geometric engineering and mirror symmetry

When  $\Gamma$  is a cyclic group,  $Y^\Gamma$  is a toric variety and it admits a family of mirror spectral curves  $(\mathcal{C}^\Gamma, \Omega^\Gamma)$  described by Hori–Iqbal–Vafa [HIV00, BGST10]. When  $\Gamma$  is non-abelian,  $Y^\Gamma$  is not toric anymore as the fibers in the ALE fibration only possess the one-dimensional torus action (3.14), which is the lift of the scalar action on  $\Gamma \backslash \mathbb{C}^2$ ; as a result the standard toric methods used to deduce an explicit picture in terms of mirror Calabi–Yau 3-folds (let alone mirror curves)

<sup>7</sup>The class  $\beta \in H_2(Y^\Gamma)$  here is retrieved as the image of  $\zeta$  under the connecting morphism in the relative homology exact sequence for  $(Y^\Gamma, \mathcal{L}^\Gamma)$ . As the constraint  $\partial\zeta = \sum_i d_i$  for the moduli space to be non-empty singles out a unique pre-image  $\zeta$  for  $\beta$ , we slightly abuse notation and switch  $\zeta \leftrightarrow \beta$  to emphasize the dependence of  $W_{g,n}^{\text{GW}}$  on the bulk/boundary moduli.

do not apply here. However, at least in some special limits it has been argued in the physics literature that the genus zero A-topological string on  $Y^\Gamma$  should be governed by special geometry on a family of curves. Denoting by  $t_B$  and  $t_j$  the Kähler parameters of  $\delta$ ,  $\gamma_j \in H^2(Y_{\text{res}}^\Gamma, \mathbb{Z})$ , it was proposed in a series of papers [KKL<sup>+</sup>96, KLM<sup>+</sup>96, KKV97] that the  $g = 0$  free energy of the type A-topological string on  $Y_{\text{res}}^\Gamma$  should coincide with the prepotential of  $\mathcal{N} = 2$ ,  $d = 4$  pure super Yang–Mills with gauge group  $\mathcal{G}_\Gamma$  upon identifying the quantum Coulomb moduli as  $a_j = t_j/\epsilon$ , the holomorphic scale as  $\Lambda = e^{-t_B/4}/\epsilon$ , and taking the limit  $\epsilon \rightarrow 0$ . This limit corresponds to a type IIA compactification on a  $K3$  where we “zoom” around an ADE singularity by sending the Planck mass to infinity. The overall effect is to decouple the gravitational modes and give rise at the same time to enhanced ADE gauge symmetry. Further fibering that over a  $\mathbb{P}^1$  yields a pure gauge field theory in  $d = 4$  with eight supercharges and no hypermultiplets as the effective four-dimensional theory. As a result, in this degenerate situation we *do* expect a spectral curve mirror: this is the Seiberg–Witten curve of the geometrically engineered gauge theory.

What about the case of finite  $\epsilon$ ? When  $\Gamma = \mathbb{Z}/p\mathbb{Z}$ , i.e.  $\mathcal{G}_\Gamma = A_{p-1}$ , it was argued in [LN98] that uplifting the reasoning above to M-theory compactified on a circle gives rise to *exactly* the same type of identification, where now the UV scale  $1/\epsilon$  is identified with the inverse of the radius of the eleventh dimensional circle. This gives an exact identification of the gauge theory prepotential of the resulting  $\mathcal{N} = 1$ ,  $d = 5$  field theory with the topological string free energy: the “field theory limit” of [KKL<sup>+</sup>96, KLM<sup>+</sup>96, KKV97] becomes here just the limit from five to four dimensions. The upshot is that the sought-for mirror of  $Y^{\mathbb{Z}/p\mathbb{Z}}$  should take the form of a  $d = 5$  Seiberg–Witten curve for the pure gauge theory with group  $\mathcal{G}_\Gamma$ . When  $\mathcal{G}_\Gamma = A_{p-1}$ , this was obtained by Nekrasov in [Nek98], and the resulting geometry is the spectral curve  $\mathcal{C}_{A_{p-1}}^{\text{SW}}$  of the periodic relativistic Toda chain with  $p$ -particles [Rui90]:

$$\mathcal{C}_{A_{p-1}}^{\text{SW}} = \left\{ (X, Y) \in \mathbb{C}^* \times \mathbb{C}^*, \quad e^{-t_B/2} (X + X^{-1}Y^p) = Y^p + \sum_{k=1}^{p-1} u_{p-k}(-Y)^k + 1 \right\}, \quad (3.24)$$

equipped with the 1-form:

$$\Omega_{A_{p-1}}^{\text{SW}} = \log Y \frac{dX}{X}. \quad (3.25)$$

Unsurprisingly, this coincides with the Hori–Iqbal–Vafa mirror of  $Y^{\mathbb{Z}/p\mathbb{Z}}$ . Using brane constructions, Nekrasov’s result has been generalized to arbitrary classical groups, and in particular  $\mathcal{G} = D_{p+2}$  in [BIS<sup>+</sup>97]:

$$\mathcal{C}_{D_{p+2}}^{\text{SW}} = \left\{ (X, Y) \in \mathbb{C}^* \times \mathbb{C}^*, \quad e^{-t_B/2} (X + X^{-1}) (Y^2 - 1)^2 Y^p = (-1)^p 2^{-2p} \prod_{j=1}^{p+2} (Y - r_j)(Y - r_j^{-1}) \right\}, \quad (3.26)$$

again with the canonical Seiberg–Witten differential  $\Omega_{D_{p+2}}^{\text{SW}} = \log Y dX/X$ .

No results are available in the literature for the exceptional cases away from the  $4d$  limit (see however [LW98, EWY01] for the  $E_6$  and  $E_7$  cases when  $\epsilon \rightarrow 0$ ). However, Nekrasov’s original insight [Nek98] naturally suggests that the resulting geometry should be in all cases the spectral curve of a relativistic deformation of the Lie-algebraic Toda systems relevant for the four-dimensional

limit [MW96]. Fortunately, the relevant technology for the construction of the spectral curves has recently become available since the work of Williams [Wil13] and Fock–Marshakov [FM97, FM14], as we now turn to review.

### 3.6 B-model: the classical affine co-extended ADE Toda chain

A simple, simply-laced Lie group  $\mathcal{G}$  of rank  $R$ , with maximal torus  $\mathcal{T}$ , can be endowed with a canonical Drinfeld–Jimbo Poisson structure

$$\{g \circledast g\} = -\frac{1}{2}[r, g \otimes g], \quad (3.27)$$

where

$$r = \sum_{\alpha \in \Delta^+} e_\alpha \otimes e_{\bar{\alpha}} + \frac{1}{2} \sum_{i=1}^R h_i \otimes h_i \quad (3.28)$$

is the canonical solution of the classical Yang–Baxter equation on  $\mathcal{G}$  [KS97]; here  $\Delta_+$  is the set of positive roots, and  $(h_i, e_\alpha, e_{\bar{\alpha}})$  is a Chevalley basis of generators of  $\text{Lie}(\mathcal{G})$ . We choose a labeling of the nodes of the Dynkin diagram of  $\mathcal{G}$  by  $i = 1, \dots, R$ , which leads in turn to a labeling of the Cartan generators.  $\mathcal{G}$  has a cell-decomposition

$$\mathcal{G} = \coprod_{w \in \mathfrak{W}_{\mathcal{G}} \times \mathfrak{W}_{\mathcal{G}}} \mathcal{G}_w, \quad (3.29)$$

where  $\mathfrak{W}_{\mathcal{G}}$  is the Weyl group of  $\mathcal{G}$  and the double Bruhat cells  $\mathcal{G}_w$  are themselves Poisson manifolds. As  $\mathcal{T} \subset \mathcal{G}$  is a trivial Poisson subgroup of  $\mathcal{G}$ , the Poisson structure (3.27) descends to Poisson structures on  $\mathcal{G}/\mathcal{T}$  and  $\mathcal{G}_w/\mathcal{T}$ , where the quotient is taken by the adjoint action of the torus. Given a standard decomposition of a word  $w \in \mathfrak{W}_{\mathcal{G}} \times \mathfrak{W}_{\mathcal{G}}$  of length  $l$  into reflections  $w = \psi_{i_1} \circ \dots \circ \psi_{i_l}$  with respect to the simple roots  $\alpha_{i_j}$  labeled by the nodes  $i_j$  of the Dynkin diagram, the map

$$L_w^{\mathcal{G}} : \begin{array}{ccc} (\mathbb{C}^*)^l & \longrightarrow & \mathcal{G}_w/\mathcal{T} \\ \{\varkappa_m\}_{m=1}^l & \longrightarrow & \prod_{m=1}^l H_{i_m}(\varkappa_m) E_{i_m} \end{array} \quad (3.30)$$

is a Poisson morphism with respect to the logarithmically constant Poisson structure on  $(\mathbb{C}^*)^l$  determined by the exchange matrix  $\epsilon$  on the corresponding Poisson quiver (see [KM15]):

$$\{\varkappa_i, \varkappa_j\} = \epsilon_{ij} \varkappa_i \varkappa_j. \quad (3.31)$$

In (3.30),  $H_i(\varkappa) = \exp(\varkappa h_i)$  and  $E_i = \exp(e_i)$  are elements of  $\mathcal{G}$  obtained by exponentiating the Chevalley generators. The operator  $L_w^{\mathcal{G}}$  is the Lax matrix of a classical integrable system on  $\mathcal{G}_w/\mathcal{T}$ : the coefficients of its characteristic polynomial give then a set of independent Ad-invariant (hence Poisson commuting) functions on  $\mathcal{G}_w/\mathcal{T}$ .

When  $\mathcal{G} = \text{SL}(p+1)$ , the resulting mechanical system is the open relativistic Toda chain with  $p$  sites [Rui90]. As was the case for the Lie-algebraic version of the non-relativistic Toda system, generalizing this picture to the periodic case relevant for the discussion of the previous section amounts to extending the construction above to the case of affine Lie groups. It was proposed in [FM14] that the relevant Poisson submanifolds in this case should be constructed on the co-extended loop group  $\mathcal{G}^{\#} \simeq \text{Loop}(\mathcal{G}) \rtimes \mathbb{C}^*$ , upon projecting onto elements having trivial

co-extension. In particular, we focus on the double Bruhat cell labeled by the cyclically irreducible word:

$$\mathbf{w} \triangleq 1\bar{1} \dots R\bar{R}. \quad (3.32)$$

The corresponding Lax matrix  $L_{\mathbf{w}}^{\mathcal{G}^\#}$  is obtained from  $L_{\mathbf{w}}^{\mathcal{G}}$  by adjoining a spectral parameter-dependent contribution by the affine root of  $\text{Loop}(\mathcal{G})$  [KM15], as

$$L_{\mathbf{w}}^{\mathcal{G}^\#}(\varkappa_1, \varkappa_{\bar{1}}, \dots, \varkappa_R, \varkappa_{\bar{R}}; X) \triangleq \prod_{i=1}^R H_i(\varkappa_i) E_i H_i(\varkappa_i) E_{\bar{i}} E_0(X/\varkappa_0) E_{\bar{0}}(X^{-1}), \quad (3.33)$$

with  $\varkappa_0, X \in \mathbb{C}^*$  and the product is done starting from  $i = 1$  on the left and ending at  $i = R$  on the right. Denote by  $(\chi_{\omega_i})_{i=1}^R$  the characters of the fundamental representation with highest weight  $\omega_i$ , where  $\omega_i(\alpha_j) = \delta_{ij}$ . We have a map

$$\begin{aligned} u : (\mathbb{C}^*)^{2R} \times \mathbb{C}_{\varkappa_0}^* \times \mathbb{C}_X^* &\longrightarrow \mathbb{C}_u^R \\ L_{\mathbf{w}}^{\mathcal{G}^\#} &\longmapsto \chi_{\omega_i}(L_{\mathbf{w}}^{\mathcal{G}^\#, [0]}) \end{aligned} \quad (3.34)$$

obtained by taking the constant term  $L_{\mathbf{w}}^{\mathcal{G}^\#, [0]}$  in the Laurent expansion of  $L_{\mathbf{w}}^{\mathcal{G}^\#} \in \mathcal{G}[X, X^{-1}]$  and then evaluating its fundamental characters. This is a submersion of  $(\mathbb{C}^*)^{2R+2}$  onto a Zariski open subset  $\mathcal{U}_{\mathcal{G}}$  of  $\mathbb{C}^R$  with the linear coordinates:

$$u_i = \chi_{\omega_i}(L_{\mathbf{w}}^{\mathcal{G}^\#, [0]}) \quad (3.35)$$

giving a complete set of hamiltonians in involution. Furthermore, let  $l_i \in \mathbb{N}$  be the coefficients of the highest positive root in the  $\alpha$ -basis for  $\mathcal{G}$ . Then, upon projecting to trivial co-extension,

$$u_0 \triangleq \varkappa_0^{1/2} \prod_{i=1}^R \varkappa_i^{l_i} = \varkappa_0^{-1/2} \prod_{i=1}^R \varkappa_{\bar{i}}^{-l_i} \quad (3.36)$$

gives a Casimir for the Poisson bracket on  $\mathcal{G}_{\mathbf{w}}^\#$ . Fix now an arbitrary irreducible representation  $\rho \in \text{Rep}(\mathcal{G})$ . The characteristic polynomial of  $\rho(L_{\mathbf{w}}^{\mathcal{G}^\#})$  then gives a family of plane curves  $\mathcal{C}_{\mathcal{G}^\#}^{\text{Toda}} \subset (\mathbb{C}^*)^2$  over  $\hat{\mathcal{U}}_{\mathcal{G}} \triangleq \mathbb{C}_{u_0}^* \times \mathcal{U}_{\mathcal{G}}$ . The curve above a point  $u = (u_0, \dots, u_R)$  given by

$$\mathcal{C}_{\mathcal{G}^\#, \rho}^{\text{Toda}} = \{(X, Y) \in \mathbb{C}^* \times \mathbb{C}^*, \quad \det[Y\mathbf{1} - \rho(L_{\mathbf{w}}^{\mathcal{G}}(\varkappa; X))] = 0\}. \quad (3.37)$$

We further equip  $\mathcal{C}_{\mathcal{G}^\#, \rho}^{\text{Toda}}$  with the 1-form:

$$\Omega_{\mathcal{G}^\#, \rho}^{\text{Toda}} = \log Y \frac{dX}{X}. \quad (3.38)$$

When  $\mathcal{G} = A_1 = \text{SL}(2)$  and  $\rho = \square$  is the fundamental representation, this is just the holomorphic Poincaré 1-form on the phase space of the relativistic Toda particle.

## 4 The two main conjectures

It can easily be shown that, upon specializing (3.37) to  $(\mathcal{G} = A_p, \rho = \square)$  and  $(\mathcal{G} = D_{p+2}, \rho = \mathbf{2}(\mathbf{p} + \mathbf{2})_{\mathbf{v}})$ , we obtain [Nek98, KM15] that:

$$\mathcal{C}_{A_p, \square}^{\text{Toda}} = \mathcal{C}_{A_p}^{\text{SW}}, \quad \text{and} \quad \mathcal{C}_{D_{p+2}, \mathbf{2}(\mathbf{p} + \mathbf{2})_{\mathbf{v}}}^{\text{Toda}} = \mathcal{C}_{D_{p+2}}^{\text{SW}} \quad (4.1)$$

after suitably identifying the action variables  $u = (u_0, \dots, u_R)$  in (3.37) with the classical Coulomb vacuum expectation values in (3.24)-(3.26). This compels us to formulate the two following conjectures.

Let  $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$  be an isometry group of  $\mathbb{S}^3$  isomorphic to a cyclic or binary polyhedral group  $\Gamma$ . Let  $\mathcal{D}_\Gamma$  its Dynkin diagram determined by McKay correspondence (Table 1), and  $\mathcal{G}_\Gamma$  the associated simply connected, simply-laced Lie group. We specialize  $\rho_{\min}$  to be an irreducible  $\mathcal{G}_\Gamma$ -module of minimal dimension, as in the following table (we will comment on non-minimal representation at the end of this Section). We can thus abbreviate  $\mathcal{C}_{\mathcal{G}^\#}^{\mathrm{Toda}, \rho_{\min}} \triangleq \mathcal{C}_{\mathcal{G}^\#}^{\mathrm{Toda}}$ , and denote the family  $\psi : \mathcal{C}_{\mathcal{G}^\#}^{\mathrm{Toda}} \rightarrow \mathfrak{U}_{\mathcal{G}}$ .

$\mathcal{G}_\Gamma$	$\rho_{\min}$
$A_{p-1}$	$\square, \square$
$D_4$	$8_v, 8_s, 8_c$
$D_{p+2}$	$2(p+2)_v, p > 2$
$E_6$	$27, \overline{27}$
$E_7$	$56$
$E_8$	$248$

Table 3: Minimal irreducible modules for the ADE Lie groups.

The first conjecture states that, upon suitable restriction of the action variables in  $\mathfrak{U}_{\mathcal{G}_\Gamma}$  and quantum cohomology parameters of  $Y^\Gamma$ , the (affine co-extended) Toda spectral curves are a subfamily of mirror curves of  $Y^\Gamma$  that coincides with the LMO spectral curves of  $\mathbb{S}^\Gamma$ . Here, the only place where the Seifert invariant  $\sigma$  appears is in the rescaling  $\lambda = \hat{\lambda}/\sigma$  of the string coupling constant. Recall that  $c = \exp(\chi_{\mathrm{orb}}\lambda/2a)$ .

**Conjecture 4.1** (a) *There exists a family of curves  $\phi : \tilde{\mathcal{C}}_{\mathcal{D}_\Gamma}^{\mathrm{LMO}} \rightarrow \mathfrak{T}_{\mathcal{D}_\Gamma}^{\mathrm{LMO}}$  over a 1-dimensional base, and a finite surjective map  $\kappa : \mathfrak{T}_{\mathcal{D}_\Gamma}^{\mathrm{LMO}} \rightarrow \mathbb{C}^*$ , such that the germs at  $c = 1$  of the LMO spectral curve and of  $\kappa \circ \phi$  are canonically isomorphic.*

(b) *The base  $\mathfrak{T}_{\mathcal{D}_\Gamma}^{\mathrm{LMO}}$  is isomorphic to  $\mathbb{A}^1$ .*

(c) *We have a commutative diagram:*

$$\begin{array}{ccc}
 \tilde{\mathcal{C}}_{\mathcal{D}_\Gamma}^{\mathrm{LMO}} & \xrightarrow{\vartheta} & \mathcal{C}_{\mathcal{G}^\#}^{\mathrm{Toda}} \\
 \phi \downarrow & & \downarrow \psi \\
 \mathfrak{T}_{\mathcal{D}_\Gamma}^{\mathrm{LMO}} & \xrightarrow{\theta} & \mathfrak{U}_{\mathcal{G}_\Gamma}
 \end{array}$$

where  $\theta$  is a finite immersion and  $\vartheta$  restricted to any fiber is an isomorphism.

(d) *There exists a choice  $t \leftarrow t(\lambda)$  of quantum cohomology parameters such that the generating series  $F_g$  (resp.  $W_{g,n}$ ) computed by the topological recursion to the restricted subfamily*

$\mathcal{C}_{\mathcal{G}^\#}^{\text{Toda}}|_{\text{Im } \theta}$  coincide with the genus- $g$  closed (resp.  $n$ -holes, open) Gromov–Witten potential of the 3-fold geometry  $(Y^\Gamma, \mathcal{L}^\Gamma)$  described in (3.15), (3.21). Up to symplectic transformations of  $(X, Y)$  and overall multiplication by a constant, the 1-form to use as input of the recursion is  $\Omega_{\mathcal{G}^\#}^{\text{Toda}} = \ln Y \, dX/X$  restricted to  $\text{Im } \theta$ .

We formulate a second conjecture, extending the previous one to generic action variables/generic vacua in Chern–Simons theory. Since  $\pi_1(\mathbb{S}^\Gamma) = \Gamma$ , the set of critical points of the Chern–Simons action is:

$$\mathfrak{V}_{\Gamma, N} \triangleq \{ \text{flat } \text{U}(N) \text{ connections on } \mathbb{S}^\Gamma \text{ modulo gauge} \} \simeq \text{Hom}(\Gamma, \text{U}(N))/\text{U}(N). \quad (4.2)$$

and we let  $\mathfrak{V}_\Gamma = \lim_{N \rightarrow \infty} \mathfrak{V}_{\Gamma, N}$  be its direct limit with respect to the composition of morphisms given by the embedding  $\text{U}(N) \hookrightarrow \text{U}(N+1)$ . By the McKay correspondence [McK80], irreducible representations of  $\Gamma$  are labeled by the nodes of the extended Dynkin diagram  $\tilde{\mathcal{D}}_\Gamma$ . The affine node labels the trivial representation, and for  $i \geq 1$ , these dimensions coincide with the components of the highest root of  $\mathcal{G}_\Gamma$  in the basis of simple roots. We can then describe:

$$\mathfrak{V}_\Gamma = \mathbb{N}^{R+1}, \quad \mathfrak{V}_{\Gamma, N} = \left\{ (N_0, \dots, N_R) \in \mathbb{N}^{R+1}, \quad N_0 + \sum_{i=1}^R D_i N_i = N \right\}. \quad (4.3)$$

When  $N \rightarrow \infty$ , we consider a background  $[\mathcal{A}]_{\mathfrak{t}}$  parametrized by  $\mathfrak{t}_i \triangleq N_i \hbar$  for  $i \in \llbracket 0, R \rrbracket$ , and in particular the rank is encoded in  $\hat{\lambda} = \mathfrak{t}_0 = N \hbar$ . We also define  $\mathfrak{c}_i = \exp(\chi_{\text{orb}} \mathfrak{t}_i / 2a)$ . Let now  $\mathcal{F}_g(\mathbb{S}^\Gamma, \mathfrak{t})$  and  $\mathcal{W}_{g,n}(\mathbb{S}^\Gamma, \mathfrak{t}; \vec{x})$  be the perturbative free energies and correlators of  $\text{U}(N)$  Chern–Simons theory expanded around the background  $[\mathcal{A}]_{\mathfrak{t}}$ , which is defined at least formally as a series in  $\mathfrak{t}$  by the ribbon graph expansion of Section 3.1. While it is not clear to us if this can be given a matrix model-like expression beyond the A-series, e.g. by collecting certain terms in the exact Chern–Simons partition functions derived in [Mar04, BT13], the spectral curve in the background  $[\mathcal{A}]_{\mathfrak{t}}$  can nevertheless be defined as in (2.9) from  $\mathcal{W}_{0,1}$ , and it yields a family of curves  $\phi_0 : \mathcal{C}_{\mathcal{D}_\Gamma}^{\text{CS}} \rightarrow (\mathbb{C}_{\mathfrak{t}}^{R+1})_{\text{formal}}$  where the notation for the base means that it is a priori a formal neighborhood of 0 in  $\mathbb{C}^{R+1}$ . In light of the previous remark, we are unable to propose an independent computation for this Chern–Simons spectral curve in a general background, but we speculate:

**Conjecture 4.2** (a) *There exists a family of curves  $\phi : \tilde{\mathcal{C}}_{\mathcal{D}_\Gamma}^{\text{CS}} \rightarrow \mathfrak{T}$  over an  $(R+1)$ -dimensional base, and a finite surjective map  $\kappa : \mathfrak{T}_{\mathcal{D}_\Gamma} \rightarrow \prod_{i=0}^R \mathbb{C}_{\mathfrak{c}_i}^*$ , such that  $\phi_0$  and the germ at  $c=1$  (i.e.  $\mathfrak{t}=0$ ) of  $\kappa \circ \phi$  are canonically isomorphic.*

(b) *We have a commutative diagram:*

$$\begin{array}{ccc} \tilde{\mathcal{C}}_{\mathcal{D}_\Gamma}^{\text{CS}} & \xrightarrow{\vartheta} & \mathcal{C}_{\mathcal{G}^\#}^{\text{Toda}} \\ \phi \downarrow & & \downarrow \psi \\ \mathfrak{T}_{\mathcal{D}_\Gamma} & \xrightarrow{\theta} & \mathfrak{U}_{\mathcal{G}_\Gamma} \end{array}$$

where  $\theta$  is a finite map and  $\vartheta$  is a fiberwise isomorphism.

(c) There exists a section  $\tilde{\mathbf{t}}$  (the orbifold mirror map) of  $\theta$  such that the topological recursion applied to  $\mathcal{C}_{\mathcal{G}_\Gamma}^{\text{Toda}}$  and the 1-form  $\ln Y \, dX/X$  (maybe up to rescaling by a constant) computes, above the point  $u$ , the free energy  $\mathcal{F}_g(\mathbb{S}^\Gamma, \tilde{\mathbf{t}}(u))$  and the correlators  $W_{g,n}(\mathbb{S}^\Gamma, \tilde{\mathbf{t}}(u); \vec{x})$ , with  $x = X^{1/a}$ .

(d) There exists an affine automorphism  $\ell_\Gamma \in \mathbb{C}^{r_{\mathcal{G}_\Gamma}} \rtimes \text{End}(\mathbb{C}^{r_{\mathcal{G}_\Gamma}})$  such that

$$\mathcal{F}_g^{\text{CS}}(\mathbb{S}^\Gamma, \mathbf{t}) = F_g^{\text{GW}}(Y_{\text{orb}}^\Gamma, \ell_\Gamma(t)), \quad \mathcal{W}_{g,h}^{\text{CS}}(\mathbb{S}^\Gamma, \mathbf{t}, \vec{x}) = W_{g,n}^{\text{GW}}(Y_{\text{orb}}^\Gamma, \mathcal{L}_{\text{orb}}^\Gamma, \ell_\Gamma(t); \vec{x}). \quad (4.4)$$

Furthermore, there exists a unique change of normalization of the periods of  $\mathcal{W}_{0,2}^{\text{CS}}$  (call it  $\widetilde{\mathcal{W}}_{0,2}^{\text{CS}}$ ) such that the ensuing topological recursion on  $\widetilde{\mathcal{C}}_{\mathcal{D}_\Gamma}^{\text{CS}}$  gives generating functions  $\widetilde{\mathcal{F}}_g^{\text{CS}}, \widetilde{\mathcal{W}}_{g,n}^{\text{CS}}$  with

$$\widetilde{\mathcal{F}}_g^{\text{CS}}(\mathbb{S}^\Gamma, \mathbf{t}) = F_g^{\text{GW}}(Y_{\text{res}}^\Gamma, \ell_\Gamma(t)), \quad \widetilde{\mathcal{W}}_{g,n}^{\text{CS}}(\mathbb{S}^\Gamma, \mathbf{t}, \vec{x}) = W_{g,n}^{\text{GW}}(Y_{\text{res}}^\Gamma, \mathcal{L}_{\text{res}}^\Gamma, \ell_\Gamma(t); \vec{x}). \quad (4.5)$$

**Remark 4.1** (On minimal orbits and minimal irreps). The construction of the Toda spectral curve involves the choice of a minimal-dimensional representation  $\rho_{\min}$  of  $\mathcal{G}_\Gamma$ ; picking up a different representation leads to a curve, which cannot be simply reconstructed from the minimal one, but that should however lead to the same free energies [MW96]. On the other hand, we will see in Section 5.1 that the construction of the LMO spectral curve likewise depends on the choice of a vector  $v \in \mathbb{Z}^a$  with finite orbit under a monodromy group  $\mathfrak{W}' = \text{Weyl}(\mathcal{D}'_\Gamma)$  for a certain  $\mathcal{D}'_\Gamma \subseteq \mathcal{D}_\Gamma$ : different choices of  $v$  contain equivalent information which is just repackaged differently, though in a non-trivial way, since the degree of the curves is related to the size of the orbit of  $v$ . One may wonder if there is a set-theoretic injection of the set of finite monodromy orbits into  $\text{Rep}(\mathcal{G}_\Gamma)$ , and whether the higher degree curves on the LMO side should be obtained from (suitable restrictions of) non-minimal Toda spectral curves.

**Remark 4.2** (Central extensions of  $\Gamma$ ). A finite isometry subgroup  $\tilde{\Gamma} \subset \text{SO}(4)$  of  $\mathbb{S}^3$  is generically a non-trivial central extension of one of the finite groups  $\Gamma$  of Table 1 (see Appendix A for more details). The reasoning of Section 3.4 would lead us to consider now ALE fibrations over the weighted projective line, as in this case the  $\tilde{\Gamma}$ -action on the resolved conifold acts effectively on the base  $\mathbb{P}^1$ . It was shown by one of the authors in [BGST10] for the  $A$ -series that the geometric transition argument cannot be applied verbatim in this setting. We leave this question to future investigations.

**Remark 4.3** ( $\Gamma$ -action and orientifolds). In our brane construction of Section 3.4, if we instead chose  $\sigma(a_{11}) = \overline{a_{22}}$ ,  $\sigma(a_{12}) = -\overline{a_{21}}$  as our anti-holomorphic involution, we would have that  $Y_\sigma = \emptyset$ : this would correspond to the orientifold of the resolved conifold considered by Sinha and Vafa in [SV00], and in turn to Chern–Simons theory on  $\mathbb{S}^3$  with  $\text{SO}/\text{Sp}$  gauge group at large  $N$ . In contrast with the discussion of Section 3.4, it is straightforward to check that in this case the  $\Gamma$ -action commutes with the real involution for all finite  $\Gamma \subset \text{SU}(2)$ . In particular, open and closed *real* versions of the Gromov–Witten potentials (3.22) and (3.23) can be defined by unoriented localization, as in [BFM05, DFM03]. On the other hand,  $\text{SO}/\text{Sp}$  Chern–Simons invariants of  $\mathbb{S}^\Gamma$  can also be computed from a matrix model analysis and the topological recursion [BE14, Section 8]: the spectral curve and two-point function is the same as for  $\text{SU}$  up to a renormalization  $\lambda \rightarrow \lambda/2$ , but the initial data is enriched by an (explicit) 1-point genus 1/2 function. It is possible to formulate the analogue of Conjectures 4.1–4.2 in this context, but we will not venture in collecting supporting evidence here.

## 5 Computations I: the LMO curves

### 5.1 LMO spectral curves

The LMO spectral curve is characterized as a solution of a maximization problem, which can be presented in several ways. In terms of the large  $N$  spectral density  $\varrho(\phi)$  for the  $\phi_i$ 's, we have the saddle point equation:

$$\oint \varrho(\phi') \left\{ (2-r) \ln \sinh[(\phi - \phi')/2] + \sum_{m=1}^r \ln \sinh[(\phi - \phi')/2a_m] \right\} \leq \frac{\phi^2}{2\lambda}, \quad (5.1)$$



with equality on the support of  $\varrho$ , and  $\varrho \geq 0$  with total mass  $\int \varrho(\phi) d\phi = 1$ . When  $\chi_{\text{orb}} > 0$ , one can show that the solution of this problem is unique, the support is a segment  $S$ , and  $\varrho(\phi)$  is of the form  $\mathbf{1}_S(\phi) \sqrt{Q(\phi)}$  with  $Q$  a positive, real-analytic function vanishing at the endpoints of  $S$ . Given the symmetry  $\{\phi_i \rightarrow -\phi_i, \quad 1 \leq i \leq N\}$  of the model (2.8),  $S$  must be symmetric around 0. Its determination is part of the problem. The usual method is to solve the linear equation (5.1) for a fixed arbitrary segment, then list the possible segments compatible with the other constraints (total mass 1, vanishing of the  $\varrho$  at the edges). This list is usually finite, and if there is not already a unique solution, the correct one is singled out by the positivity constraint  $\varrho \geq 0$ . However, it is by no means easy to solve explicitly singular integral equations of the form (5.1) on a segment.

The linear equation (5.1) can be rewritten in several equivalent forms. In terms of the resolvent:

$$W(x) \triangleq \int \frac{x \varrho(\phi) d\phi}{x - e^{\phi/a}}, \quad \varrho(\phi) \triangleq \frac{W(e^{\phi/a} - i0) - W(e^{\phi/a} + i0)}{2i\pi e^{\phi/a}}, \quad (5.2)$$

it becomes, for all  $x \in S$ :

$$W(x + i0) + W(x - i0) + (2 - r) \sum_{\ell=1}^{a-1} W(\zeta_a^\ell x) + \sum_{m=1}^r \sum_{\ell_m=1}^{a/a_m-1} W(\zeta_{a/a_m}^{\ell_m} x) = (a^2/\lambda) \ln x + (a/2) \chi_{\text{orb}} \quad (5.3)$$

where  $\zeta_k$  is a primitive  $k$ -th root of unity. The symmetry  $\{\phi_i \rightarrow -\phi_i \quad 1 \leq i \leq N\}$  implies:

$$W(x) + W(1/x) = 1. \quad (5.4)$$

We can get rid of the right-hand side and of log-singularities by defining:

$$\mathcal{Y}(x) \triangleq -cx \exp[(\chi_{\text{orb}} \lambda / a) W(x)], \quad c \triangleq \exp(\chi_{\text{orb}} \lambda / 2a). \quad (5.5)$$

By construction,  $\mathcal{Y}(x)$  is a holomorphic function on  $\mathbb{C} \setminus S$ , with behavior

$$\mathcal{Y}(x) \underset{x \rightarrow 0}{\sim} -cx, \quad \mathcal{Y}(x) \underset{x \rightarrow \infty}{\sim} -c^{-1}x \quad (5.6)$$

and satisfying:

$$\forall x \in S, \quad \mathcal{Y}(x + i0) \mathcal{Y}(x - i0) \left[ \prod_{\ell=1}^{a-1} \mathcal{Y}(\zeta_a^\ell x) \right]^{2-r} \cdot \prod_{m=1}^r \left[ \prod_{\ell_m=1}^{a/a_m-1} \mathcal{Y}(\zeta_{a/a_m}^{\ell_m} x) \right] = 1. \quad (5.7)$$

The symmetry (5.4) becomes:

$$\mathcal{Y}(x) \mathcal{Y}(1/x) = 1. \quad (5.8)$$

Equation (5.6) can be seen as a description of generators for the monodromy group  $\mathfrak{G}$  of the analytic function  $\mathcal{Y}(x)$ , and (5.6) are constraints imposed on the singularities of the solution away from the branchcuts (here meromorphic singularities at 0 and  $\infty$ ). [BE14] presented a general strategy to solve a class of monodromy equations including (5.7). It leads at least to a partially explicit solution when certain finite subgroups of  $\mathfrak{G}$  are identified, as one can then express the solution in terms of an algebraic function. Among the Seifert matrix model, the ADE cases turned out to be very special, because they are the ones that can be obtained from algebraic curves.

**Theorem 5.1** [BE14] Consider the equation (5.7) with  $(a_1, \dots, a_r)$  arbitrary positive integers and  $\chi_{\text{orb}} \neq 0$ .  $\mathfrak{G}$  is infinite, except in the case  $(2, 2, 2p')$  where  $\mathfrak{G}$  is the symmetric group in  $2p' + 1$  elements. Besides, the two following points are equivalent:

- (i)  $\chi_{\text{orb}} > 0$ .
- (ii) There exists a non-zero  $v \in \mathbb{Z}^a$  such that

$$\mathcal{Y}_v(x) \triangleq \prod_{j=0}^{a-1} [\mathcal{Y}(\zeta_a^j x)]^{v_j} \quad (5.9)$$

has a finite monodromy group. So, there is a polynomial  $\mathcal{P}_v$  in two variables, depending on  $\lambda$ , such that  $\mathcal{P}_v(x, \mathcal{Y}_v(x)) = 0$ .

While  $\mathcal{Y}(x)$  has a cut on  $S$  only,  $\mathcal{Y}_v(x)$  has a branchcut on  $S_j = \zeta_a^{-j} S$  whenever  $v_j \neq 0$ . Knowing  $\mathcal{Y}_v(x)$  is enough to retrieve  $W(x)$  since:

$$v_j (W(\zeta_a^j x) - 1) = \frac{a}{\chi_{\text{orb}} \lambda} \oint_{S_j} \frac{d\xi}{2i\pi} \frac{\ln(\mathcal{Y}_v(\xi)/(-c\xi))}{x - \xi}. \quad (5.10)$$

A simple computation from (5.7) shows that there exists linear involutions  $T_j \in \text{GL}(a, \mathbb{Z})$  describing the monodromy of these new functions:

$$\forall v \in \mathbb{Z}^a, \forall x \in S_j, \quad \mathcal{Y}_v(x + i0) = \mathcal{Y}_{T_j(v)}(x - i0). \quad (5.11)$$

The monodromy group  $\mathfrak{G}$  is isomorphic to the linear subgroup generated by the  $T_j$  for  $j = 0, \dots, (a-1)$ . Lemma 5.1 is then an answer to the question: does there exist a non-zero vector  $v$  with finite  $\mathfrak{G}$ -orbit? The answer is positive only for the ADE cases, and we can actually be more precise: there exists a decomposition in two lattices  $\mathbb{Z}^a = E_0 \oplus E$ , where  $E$  is stable under  $\mathfrak{G}$ , and the group generated by  $T_j|_E$  for  $j = 0, \dots, (a-1)$  is conjugate to the Weyl group  $\mathfrak{W}'$  of a finite root system. The latter are also classified by Dynkin diagrams  $\mathcal{D}'$  of ADE type, and it turns out that  $\mathcal{D}'$  is always a sub-diagram of the Dynkin diagram  $\mathcal{D}$  attached to the Seifert geometry (see Table 2), with equality only in the  $E_8$  case and certain lens spaces. Then, one can show that  $v$  has finite  $\mathfrak{G}$ -orbit iff  $v \in E$ . In that case, describing the monodromy of  $\mathcal{Y}_v(x)$  reduces to the well-known classification of the orbits of the Weyl group  $\mathfrak{W}'$  [GP00], which are in correspondence with the parabolic subgroups of  $\mathfrak{W}'$ , themselves described as the reflection groups attached to the (possibly disconnected) sub-diagrams  $\mathcal{D}''$  strictly included in  $\mathcal{D}'$ .

## 5.2 Definition of $\mathcal{P}_{\mathcal{D}}^{\text{LMO}}$

For computational purposes, it is natural to choose  $v$  in an orbit of minimal size. If  $v$  lies in a minimal orbit, all the other minimal orbits are obtained – up to rescaling – by shifting with

$$\varepsilon : (v_j)_j \mapsto (v_{j+1 \pmod{a}})_j. \quad (5.12)$$

This shift amounts to replacing  $x$  with  $\zeta_a^{-1}x$ . Minimal orbits are given in Section 5.4 for  $D$  cases, in Section 5.5 for  $E_6$  and in Appendices E.1 and F.1 for  $E_7$  and  $E_8$ . They happen to be stable under some power of the shift  $\varepsilon^{a/a'}$ , i.e.  $\mathcal{P}_v(x, y)$  is a polynomial in  $x^{a'}$ , for some  $a'$  dividing  $a$ .

- In the Seifert geometry  $\mathcal{D} = D_{2p'+2}$ ,  $v$  and  $\varepsilon[v]$  generated two disjoint minimal orbits, and  $P_v(x, y)$  is actually a polynomial in  $x^{p'}$ . Then:

$$\mathcal{P}_{D_{2p'+2}}^{\text{LMO}}(X, Y) \triangleq \mathcal{P}_v(x, Y) \mathcal{P}_v(-x, Y), \quad X = x^{2p'}. \quad (5.13)$$

- For  $\mathcal{D} = D_{2p'+3}$ , there is a unique minimal orbit (all the shifts are in the same orbit), so  $P_v(x, Y)$  is a polynomial in  $X = x^{4p'}$  that we denote  $\mathcal{P}_{D_{2p'+3}}^{\text{LMO}}(X, Y)$ .
- For  $\mathcal{D} = E_6$ , the triality of  $\mathcal{D}' = D_4$  is responsible for the existence of 3 minimal orbits, generated by  $v$ ,  $\varepsilon[v]$  and  $\varepsilon^2[v]$ , and  $\mathcal{P}_v(x, y)$  is a polynomial in  $x^2$ . Then, we introduce the polynomial:

$$\mathcal{P}_{E_6}^{\text{LMO}}(X, Y) \triangleq \mathcal{P}_v(x, Y) \mathcal{P}_v(\zeta_3 x, Y) \mathcal{P}_v(\zeta_3^{-1} x, Y), \quad X = x^6. \quad (5.14)$$

- Similarly, for  $\mathcal{D} = E_7$ , the duality of  $\mathcal{D}' = E_6$  results in the existence of 2 minimal orbits generated by  $v$  and  $\varepsilon[v]$ , and  $P_v(x, Y)$  is a polynomial in  $x^6$ . Then, we introduce the polynomial:

$$\mathcal{P}_{E_7}^{\text{LMO}}(X, Y) \triangleq \mathcal{P}_v(-x, Y) \mathcal{P}_v(x, Y), \quad X = x^{12}. \quad (5.15)$$

- For  $\mathcal{D} = E_8$ , there is a unique minimal orbit, so  $\mathcal{P}_v(x, Y)$  is a polynomial in  $X = x^a = x^{30}$ , that we denote  $\mathcal{P}_{E_8}^{\text{LMO}}(X, Y)$ .

In all cases, we have set  $X = x^a$ , and our definition for the LMO spectral curve is:

$$\mathcal{C}_{\mathcal{D}}^{\text{LMO}} = \{(X, Y) \in \mathbb{C}^* \times \mathbb{C}^*, \quad \mathcal{P}_{\mathcal{D}}^{\text{LMO}}(X, Y) = 0\}. \quad (5.16)$$

Equivalently, the ideal  $\mathcal{P}_{\mathcal{D}}^{\text{LMO}}(X, Y) = 0$  is obtained by elimination of  $x$  in the equations  $\{\mathcal{P}_v(x, Y) = 0, X = x^a\}$ . Considering  $\mathcal{P}^{\text{LMO}}$  rather than  $\mathcal{P}_v$  is necessary for comparison with the Toda spectral curves, but of course it does not contain more information than  $\mathcal{P}_v$ .

The symmetry (5.8) implies the palindromic symmetry:

$$\mathcal{P}_{\mathcal{D}}^{\text{LMO}}(X, Y) = C X^{\bullet} Y^{\bullet} \mathcal{P}_{\mathcal{D}}^{\text{LMO}}(1/X, 1/Y), \quad (5.17)$$

where  $\bullet$  are the degrees of  $\mathcal{P}_{\mathcal{D}}^{\text{LMO}}$  in the variables  $X$  and  $Y$ , given in Table 2.  $\mathcal{C}_{\mathcal{D}}^{\text{LMO}}$  is a family of spectral curve with parameter  $\lambda$ , equipped with the 1-form:

$$\tilde{\Omega} = \frac{a}{\chi_{\text{orb}} \lambda} \ln(-Y/cX) \frac{dX}{X} = \frac{a}{\chi_{\text{orb}} \lambda} \ln Y \frac{dX}{X} + df(X). \quad (5.18)$$

Adding the differential of a rational function of  $X$  does not change the free energies and correlators computed by the topological recursion, so we can equally choose the 1-form:

$$\Omega = \frac{a}{\chi_{\text{orb}} \lambda} \ln Y \frac{dX}{X}$$

Besides, the only effect of the rescaling by  $a/\chi_{\text{orb}} \lambda$  is that  $W_{g,n}$  are multiplied by  $(\chi_{\text{orb}} \lambda/a)^{2g-2+n}$ .

fiber orders	$a$	$\chi_{\text{orb}}$	$\mathcal{D}$	$\mathcal{D}'$	$\deg_X \mathcal{P}_{\mathcal{D}}^{\text{LMO}}$	$\deg_Y \mathcal{P}_{\mathcal{D}}^{\text{LMO}}$
$(p)$	$p$	$1 + 1/p$	$A_p$	$A_p$	1	$p$
$(2, 2, 2p')$	$2p'$	$1/2p'$	$D_{2p'+2}$	$A_{2p'}$	2	$2 \cdot (2p' + 1)$
$(2, 2, 2p' + 1)$	$2(2p' + 1)$	$1/(2p' + 1)$	$D_{2p'+3}$	$D_{2p'+2}$	2	$4(p' + 1)$
$(2, 3, 3)$	6	$1/6$	$E_6$	$D_4$	4	$3 \cdot 8$
$(2, 3, 4)$	12	$1/12$	$E_7$	$E_6$	6	$2 \cdot 27$
$(2, 3, 5)$	30	$1/30$	$E_8$	$E_8$	18	240

Figure 2:  $\mathcal{D}$  = Seifert geometry;  $\mathcal{D}'$  = monodromy group of the spectral curve.  $k \cdot d$  in the last column means that the reduced polynomial  $\mathcal{P}_v$  has degree  $d$  in  $y$ , and  $\mathcal{P}_{\mathcal{D}}^{\text{LMO}}$  contains  $k$  factors of  $\mathcal{P}_v$  differing by some rotations of  $x$ .

### 5.3 The computation in practice

Since the details of the orbit analysis were presented in [BE14], we focus here on the next step, i.e. the identification of the polynomial equation for the spectral curve. Denote by  $(w[i])_{i \in I}$  the list of vectors in a chosen minimal orbit generated by  $v$ , and write

$$\mathcal{P}_v(x, \mathcal{Y}_v(x)) = C \prod_{i \in I} (y - \mathcal{Y}_{w[i]}(x)) = 0 \quad (5.19)$$

for the equation of the spectral curve; the constant  $C$  will be fixed later on. Let  $\overline{\mathcal{C}}_v$  be the compact Riemann surface which is a smooth model for  $\overline{\{(x, y) \in \mathbb{C}^* \times \mathbb{C}^* : \mathcal{P}_v(x, y) = 0\}}$ . It comes with a branched covering  $x : \overline{\mathcal{C}}_v \rightarrow \mathbb{P}^1$ , and  $y$  defines a meromorphic function on  $\overline{\mathcal{C}}_v$  whose value in the  $i$ -th sheet is  $y^{(i)}(x) = \mathcal{Y}_{w[i]}(x)$ .

#### Step A

From (5.6), the functions  $y^{(i)}(x)$

$$x \rightarrow 0, \quad y^{(i)}(x) \sim (-cx)^{n_0(w[i])} \zeta_a^{n_1(w[i])}, \quad (5.20)$$

$$x \rightarrow \infty, \quad y^{(i)}(x) \sim (-x/c)^{n_0(w[i])} \zeta_a^{n_1(w[i])}, \quad (5.21)$$

with:

$$n_0(w) = \sum_{j=0}^{a-1} w_j, \quad n_1(v) = \sum_{j=0}^{a-1} j w_j. \quad (5.22)$$

This fixes the coefficients on the boundary of the Newton polygon of  $\mathcal{P}_v$  up to the overall multiplicative constant  $C$ . (5.20)-(5.21) tell us that the slopes are  $(\pm 1, n_0(w[i]))$ , therefore there exists a single monomial of degree 0 in  $y$ . We can fix  $C$  by setting this coefficient to 1. As we explained in the last paragraph, the symmetries observed by explicitly computing the orbits imply that  $\mathcal{P}_v(x, y)$  is actually a polynomial in  $x^{a'}$  with  $a'$  given in Figure 4.

In the  $D$  geometries, at this stage there is a shortcut to the final solution, reviewed in Section 5.4. In the exceptional cases, we continue and proceed by necessary conditions.

## Step B

In the solution we look for,  $y^{(i)}(x)$  derives from a single function  $W(x)$  such that  $y^{(i)}(x) = \mathcal{Y}_{w[i]}(x)$  given by formula (5.9). If we write the Taylor expansion:

$$x \rightarrow 0, \quad \frac{\chi_{\text{orb}} \lambda}{a} W(x) = \sum_{k \geq 1} \mu_k x^{k+1}, \quad (5.23)$$

we deduce that:

$$x \rightarrow 0, \quad y^{(i)}(x) = (-cx)^{n_0(w[i])} \zeta_a^{n_1(w[i])} \exp \left( \sum_{k \geq 1} \mu_{k-1} \hat{w}[i]_{k \bmod a} x^k \right), \quad (5.24)$$

where we introduced the discrete Fourier transform:

$$\hat{w}_k \triangleq \sum_{j=0}^{a-1} \zeta_a^{jk} w_j. \quad (5.25)$$

It turns out that many Fourier modes  $k \in \mathbb{Z}_a$  are zero for all vectors in the orbit of  $v$ . The set of non-zero Fourier modes  $K_{\mathcal{D}} = \bigcup_{i \in I} \{k \in \llbracket 0, a-1 \rrbracket, \quad w[i]_k \neq 0\}$  is:

- $K_{E_6} = \{1, 2, 3, 5\}$ .
- $K_{E_7} = \{3, 4, 6, 8, 9, 11\}$ .
- $K_{E_8} = \{1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 14, 16, 17, 19, 21, 22, 23, 25, 26, 27, 28, 29\}$ .

Only the  $\mu_{k-1}$ 's with  $k \in K$  will appear in the Puiseux expansion of  $y$ . Then, the sought-for polynomial takes the form:

$$\mathcal{P}_v(x, y) = B(x) \prod_{i \in I} (y - y^{(i)}(x)) \quad (5.26)$$

where the monomial prefactor  $B(x)$  is fixed by matching with the coefficient 1 of the monomial  $x^\bullet y^0$  in  $\mathcal{P}_v$ . By expanding the right-hand side of (5.26) when  $x \rightarrow 0$  using (5.24), we can express the coefficients of  $\mathcal{P}_v$  in terms of a relatively small number of  $\mu_k$  with  $k \in K_{\mathcal{D}}$ . Since we already know the Newton polygon and the symmetries of  $\mathcal{P}_v$ , we can impose those relations at the level of their expression in terms of  $\mu_k$ 's, which gives relations between the  $\mu_k$ 's and we can eliminate some of them. Doing so, we can express all coefficients of  $\mathcal{P}_v$  only in terms of  $c$  and:

- for  $E_6$ ,  $\mu_1$  and  $\mu_2$ .
- for  $E_7$ ,  $\mu_2$ ,  $\mu_3$ ,  $\mu_5$  and  $\mu_7$ .

For  $E_8$ , we could not complete this computation: it requires expanding a product of 240 factors to order  $o(x^{540})$ , and even if this would be achieved, it is still a formidable task to eliminate  $\mu_k$ 's.

## Step C

The ramification properties of the spectral curve we seek are easily described a priori. Call  $d = \deg_y \mathcal{P}_v$  the size of the orbits, i.e. the degree of  $x : \bar{\mathcal{C}}_v \rightarrow \mathbb{P}^1$ , and  $d' = \deg_x \mathcal{P}_v$  be the degree of  $y : \bar{\mathcal{C}}_v \rightarrow \mathbb{P}^1$ . By construction, the number of branchcuts of the function  $y^{(i)}(x)$  in the  $i$ -th sheet is

the number of non-zero components in  $w[i]$ , let us call it  $b[i]$ . The ramification points of  $x : \bar{\mathcal{C}}_v \rightarrow \mathbb{P}^1$  are simple and correspond to the endpoint of the branchcuts, and since each branchcut is shared by two sheets, the total number of ramification points is  $\sum_{i \in I} b[i]$ . Riemann-Hurwitz formula then gives the genus of  $\Sigma$ :

$$\text{genus}(\bar{\mathcal{C}}_v) = 1 - d + \frac{1}{2} \left( \sum_{i \in I} b[i] \right). \quad (5.27)$$

For  $E_6$ , we find that  $\bar{\mathcal{C}}_v$  has genus 5; for  $E_7$ , it has genus 46, and for  $E_8$ , it has genus 1471. We remark that the genus is much lower than the genus of a generic curve with same Newton polygon – which is the number of interior points in the Newton polygon – even if we take into account the symmetries. This means that the plane curve  $\mathcal{P}_v(x, y) = 0$  must be singular. This puts a number of algebraic constraints on the coefficients inside the Newton polygon of  $\mathcal{P}_v$ . Taking into account symmetries, we can put an upper bound on the number of independent such constraints. From our experience with the  $D$  and  $E_6$  case, we expect that implementing these constraints gives only finitely many solutions for the sequence of  $\mu_k$ . The definition in (5.23) implies that  $\mu_k = \mathcal{O}(\lambda)$  for all  $k \geq 1$  when  $\lambda \rightarrow 0$  (i.e.  $c \rightarrow 1$ ), and the  $\mu_k$ 's must have a power series expansion in  $\lambda$  with rational coefficients. We expect that this extra piece of information singles out a unique solution (among the finitely many) to the algebraic constraints.

For  $E_6$  this program is completed in Section 5.5. For  $E_7$ , performing elimination in these algebraic constraints already seem computationally hopeless, and we could not even solve them perturbatively in  $\lambda \rightarrow 0$ . Therefore, our best result is the expression of  $\mathcal{P}_v$  in terms of the  $\mu_2, \mu_3, \mu_5$  and  $\mu_7$ , which should be considered as unknown (algebraic) functions of  $c$ . This expression is given in Appendix E.2. For  $E_8$ , as we have seen, our best result is the Newton polygon and its boundary coefficients, given in Appendix F.2. Nonetheless, we will be able to compute the exact spectral curves at  $c = 1$  in all cases (see Section 6.7).

$\mathcal{D}$ (geometry)	$D_{2p'+2}$	$D_{2p'+3}$	$E_6$	$E_7$	$E_8$
$d = \deg_y \mathcal{P}_v$	$2p' + 1$	$4(p' + 1)$	8	27	240
$d' = \deg_x \mathcal{P}_v$	$4p'$	$4(2p' + 1)$	8	36	540
$a'$	$p'$	$2p' + 1$	2	6	30

Table 4: Properties of  $\mathcal{P}_v(x, y)$ .

Let us turn to the complete and explicit results that can be obtained for  $D$  and  $E_6$ .

#### 5.4 $D_{p+2}$ geometries

In that case, it is possible to guess a rational parametrization that has all the required properties, and thus gives the correct solution bypassing Steps A-B.

**$p$  even**

$v = e_0 \triangleq w[0]$  is a minimal vector, and the other vectors in the orbit are  $w[\ell] = (-1)^\ell e_\ell$  for  $\ell \in \llbracket 1, p-1 \rrbracket$ , and  $w[p] = \sum_{l=0}^{p-1} (-1)^{l+1} e_l$ . We thus get:

$\sim$	$y^{(\ell)}$	$y^{(p)}$
$x \rightarrow 0$	$\zeta_p^\ell (cx)^{(-1)^\ell}$	$-1$
$x \rightarrow \infty$	$\zeta_p^\ell (x/c)^{(-1)^\ell}$	$-1$

(5.28)

The symmetries of the problem suggest to look for a parametrization of  $\mathcal{P}_v(x, Y) = 0$  of the form:

$$\begin{cases} x(z) &= z \left( \frac{z^p - \kappa^2}{\kappa^2 z^p - 1} \right) \\ Y(z) &= - \frac{(z^{p/2} - \kappa)(\kappa z^{p/2} + 1)}{(\kappa z^{p/2} - 1)(z^{p/2} + \kappa)} \end{cases} \quad (5.29)$$

where we impose that  $z \rightarrow 0$  correspond to  $x \rightarrow 0$  in the sheet of  $w[p]$ , and  $z \rightarrow \zeta_a^\ell \kappa^{2/p}$  in the sheet of  $w[\ell]$  for  $\ell \in \llbracket 0, p-1 \rrbracket$ . Requiring (5.28), we must have:

$$\frac{2\kappa^{1+1/p}}{1 + \kappa^2} = 1/c^2 = e^{-\lambda/4p^2} \quad (5.30)$$

It can be checked that this satisfies all the desired properties of the spectral curve (including the positivity constraints), and by uniqueness, this is the solution we looked for. We can also write this parametrization with  $\zeta = z^{p/2}$  and  $X = x^a = x^p$ :

$$\begin{cases} X(\zeta) &= \zeta^2 \left( \frac{\zeta^2 - \kappa^2}{\kappa^2 \zeta^2 - 1} \right)^p \\ Y(\zeta) &= - \frac{(\zeta - \kappa)(\kappa \zeta + 1)}{(\kappa \zeta - 1)(\zeta + \kappa)} \end{cases} \quad (5.31)$$

which is now a parametrization of  $\mathcal{C}_{D_{p+2}}^{\text{LMO}}$ , for  $p$  even.

**$p$  odd**

The minimal vector is  $v \triangleq w[0] = e_0 + e_p$ , and it generates the orbit consisting in  $w[\ell] = e_\ell + e_{p+\ell}$  and  $w[p+\ell] = -w[\ell]$  for  $1 \leq \ell \leq p-1$ , and  $w[2p] = \sum_{\ell=0}^{2p-1} (-1)^\ell e_\ell$  and  $w[2p+1] = -w[2p]$ .

$\sim$	$y^{(\ell)}$	$y^{(p+\ell)}$	$y^{(2p)}$	$y^{(2p+1)}$
$x \rightarrow 0$	$-\zeta_{2p}^\ell (cx)^2$	$-\zeta_{2p}^{-\ell} (cx)^{-2}$	$-1$	$-1$
$x \rightarrow \infty$	$-\zeta_{2p}^\ell (x/c)^2$	$-\zeta_{2p}^{-\ell} (x/c)^2$	$-1$	$-1$

(5.32)

A similar guess leads to identification of the solution of  $\mathcal{P}_v(x, Y) = 0$  in parametric form:

$$\begin{cases} x^2(z) &= z^{-2} \frac{z^{2p} \kappa^2 - 1}{z^{2p} - \kappa^2} \\ Y(z) &= - \frac{(z^p - \kappa)(\kappa z^p + 1)}{(z^p \kappa - 1)(z^p + \kappa)} \end{cases} \quad (5.33)$$



Matching with the required behavior for  $x \rightarrow 0$  or  $\infty$  imposes:

$$\frac{2\kappa^{1+1/p}}{1+\kappa^2} = 1/c = e^{-\lambda/4p^2}, \quad (5.34)$$

which defines  $\kappa$  as a function of  $\lambda$  identically to (5.30). The branch has to be chosen so that:

$$\lambda \in [0, +\infty) \longleftrightarrow \kappa \in [1, 0). \quad (5.35)$$

We can also write in terms of  $\zeta = z^p$  and  $X = x^a = x^{2p}$ :

$$\begin{cases} X(\zeta) &= \zeta^{-2} \left( \frac{\zeta^2 \kappa^2 - 1}{\zeta^2 - \kappa^2} \right)^p \\ Y(\zeta) &= -\frac{(\zeta - \kappa)(\kappa \zeta + 1)}{(\zeta \kappa - 1)(\zeta + \kappa)} \end{cases} \quad (5.36)$$

which is now a parametrization of  $\mathcal{C}_{D_{p+2}}^{\text{LMO}}$ , for  $p$  odd.

### Polynomial equation

If we eliminate the variable  $\zeta$  and keep only  $X = x^a$  (which is equal to  $x^p$  if  $p$  is even and  $x^{2p}$  if  $p$  is odd) we obtain the polynomial equation  $\mathcal{P}_{D_{p+2}}^{\text{LMO}}(X, Y) = 0$  for the spectral curve. We can factor a monomial in  $\mathcal{P}_{D_{p+2}}^{\text{LMO}}(X, Y)$  to put it in the form:

$$(-1)^{p+1} e^{-\lambda/2p} (X^2 + 1)(Y^2 + 1) + XY (\kappa^2 + 1)^{-(2p+2)} \mathcal{Q}_p[(Y + 1/Y)(\kappa^2 + 1)^2] = 0, \quad (5.37)$$

where:

$$\frac{2\kappa^{1+1/p}}{\kappa^2 + 1} = e^{-\lambda/4p^2}. \quad (5.38)$$

and  $\mathcal{Q}_p(\eta) = \eta^{p+1} + \dots$  is a polynomial in  $\eta$  and  $\kappa^2$ , which does not have a uniform expression for all  $p$ 's. It is given for  $p \leq 5$  in Appendix C. These results were obtained in [BE14], where A. Weiße also checked that the solutions (5.29) and (5.33) match the results of Monte-Carlo simulations of the matrix integral (2.8) [BE14, Appendix].

## 5.5 $E_6$ geometry

After Step **A**, we arrive at a curve  $\mathcal{P}_v(x, y) = \sum_{j=0}^4 \sum_{k=0}^8 \Pi_{j,k} x^{2j} y^k$  depending on the yet unknown parameters  $\mu_1$  and  $\mu_2$ , with  $c = \exp(\lambda/72)$ .

$\Pi_{j,k}$	$j = 0$	1	2	3	4
$k = 8$			*		
7		*	*	*	
6		*	*	*	
5	*	*	*	*	*
4	$-c^{-4}$	$-2c^{-2} - 4c^{-4}\mu_1$	$4(-1 + c^{-4} + 4c^{-2}\mu_1 + 8c^{-4}\mu_1^2 + 3c^{-3}\mu_2)$	*	*
3	$-c^{-4}$	$2c^{-4}\mu_1$	$-2 + c^{-4} - 6c^{-2}\mu_1 - 4c^{-4}\mu_1^2$	*	*
2		0	$2 + 12c^{-2}\mu_1 - 6c^{-3}\mu_2$	*	
1		$c^{-2}$	$1 + 2c^{-2}\mu_1$	*	
0			1		

and the \* are the coefficients obtained by central symmetry, i.e  $\Pi_{j,k} = \Pi_{j,8-k} = \Pi_{4-j,k}$ . These expressions have been found in [BE14], and now we present a new computation.

Given the symmetry, let us define  $\xi = x^2 + 1/x^2$  and  $\eta = y + 1/y$ , and eliminate  $x$  and  $y$  from the equation  $\mathcal{P}_v(x, y) = 0$ . We obtain an equation  $\mathcal{Q}(\xi, \eta) = 0$  defining a curve of genus 2. A birational transformation  $(\xi, \eta) \mapsto (s, t)$  brings in in Weierstraß form  $s^2 = \mathcal{R}(t)$  with a polynomial of degree 5:

$$\begin{aligned} \mathcal{R}(t) = & 12(1 - c^4 - 4c^2\mu_1 - 4\mu_1^2 + 8c\mu_2)t^5 + 3(-4 + c^4 + 4c^2\mu_1 + 4\mu_1^2 - 40c\mu_2)t^4 \\ & + 24(c^3 + 2c\mu_1 + \mu_2)t^3 - 2c^2(11c^2 + \mu_2)t^2 + 8c^4t - c^4. \end{aligned} \quad (5.39)$$

Since we are looking for a singular curve,  $\mu_k$ 's should be such that the discriminant of  $\mathcal{R}$  vanishes. This discriminant is a product of two factors  $\Delta_1$  and  $\Delta_2$  given in Appendix D.1, so that gives us two equations for the two unknowns  $\mu_1$  and  $\mu_2$ , that can be solved explicitly. Among the finitely many solutions for  $(\mu_1, \mu_2)$ , there is a unique branch in which  $\mu_1 \rightarrow 0$  when  $c \rightarrow 1$ : that must be our solution. Then,  $\mu_2$  is an explicit rational function of  $\mu_1$  that we do not reproduce here, and  $\mu_1$  itself is the branch of the solution of the degree 8 equation:

$$\begin{aligned} & 256\mu_1^8 + 4864c^2\mu_1^7 + (-1024 + 35776c^4)\mu_1^6 + (62112c^2 + 125568c^6)\mu_1^5 \\ & + (1536 - 81600c^4 + 206064c^8)\mu_1^4 + (4544c^2 - 162576c^6 + 128304c^{10})\mu_1^3 \\ & + (-1024 + 55116c^4 - 78192c^8 + 26244c^{12})\mu_1^2 + (-5984c^2 + 10332c^6 - 4374c^{10})\mu_1 \\ & + 256 - 499c^4 + 243c^8 = 0, \end{aligned}$$

which behaves like  $\mu_1 = -2(c - 1) + \mathcal{O}(c - 1)^2$  when  $c \rightarrow 1$ . As a matter of fact, the solution of (5.40) has a rational uniformization. We choose the uniformizing parameter  $\kappa$  such that  $\kappa \rightarrow 0$

corresponds to  $c \rightarrow 1$ , and our final result reads:

$$\begin{aligned} \exp(\lambda/36) = c^2 &= -\frac{(\kappa-6)^4(\kappa-2)^4}{16(\kappa-3)(\kappa^2-6\kappa+12)(\kappa^2-6\kappa+6)^2}, \\ \mu_1 &= \frac{\kappa(\kappa-4)(\kappa^2-6\kappa+12)(\kappa^2-12)}{32(\kappa-3)(\kappa^2-6\kappa+6)}, \\ c \cdot \mu_2 &= -\frac{\kappa(\kappa-2)^2(\kappa-3)(\kappa-4)(\kappa-6)^2(\kappa^4-11\kappa^3+49\kappa^2-108\kappa+108)}{(\kappa^2-6\kappa+6)^4(\kappa^2-6\kappa+12)^2}. \end{aligned} \quad (5.40)$$

With these values, the spectral curve match perfectly the one computed numerically from Monte-Carlo simulations of the matrix model (2.8) by A. Weisse (Figure 3).

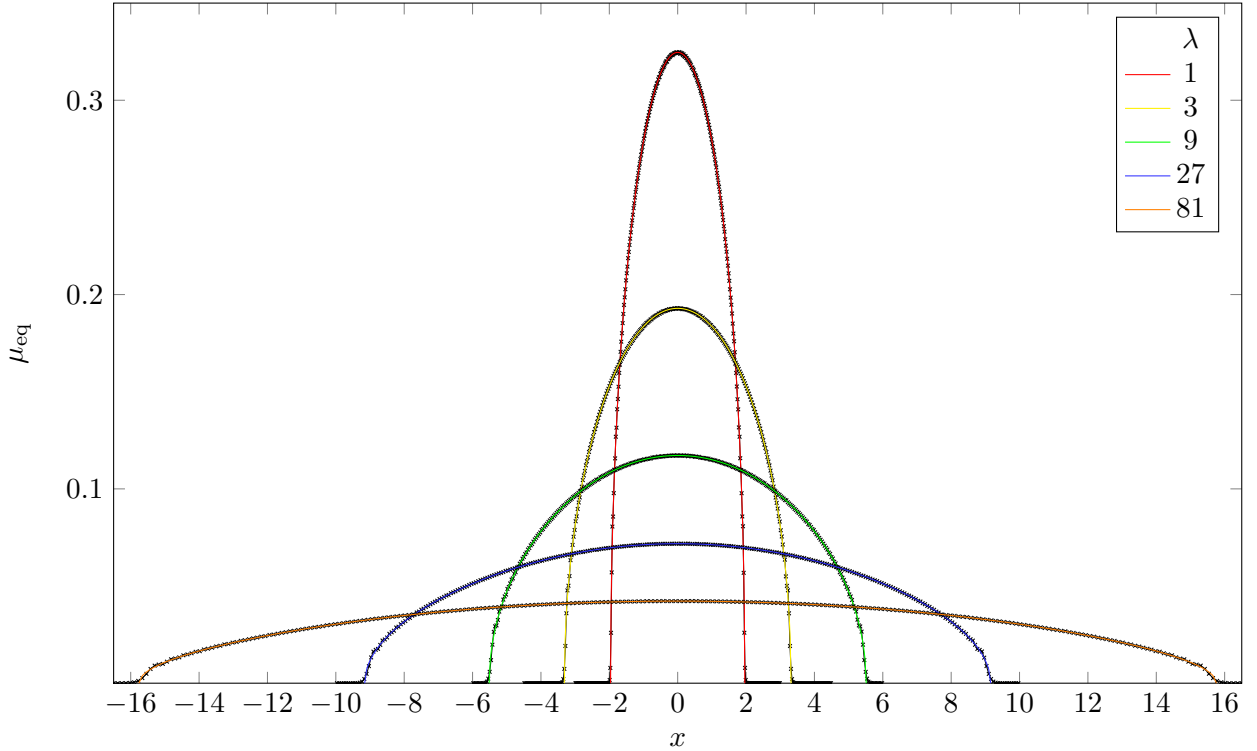


Figure 3:  $(a_1, a_2, a_3) = (2, 3, 3)$ . The dots display the Monte-Carlo simulation of the eigenvalue distribution for  $N = 200$  in the model (2.8), with various choices of  $\lambda$  given in the legend. The plain curves display the theoretical computation ensuing from the expression of  $\mathcal{P}_v(x, y) = 0$  together with (5.40).

Remarkably, the smooth model of the curve  $\mathcal{P}_v(x, y) = 0$  seen in variables  $(x^2, y)$  has genus 1, i.e. it defines an elliptic fibration over the base parameter  $\kappa \in \mathbb{P}^1$ , with discriminant:

$$\Delta(\kappa) = 2^{80} \cdot 3^4 \cdot \kappa(\kappa-2)(\kappa-3)^{16}(\kappa-4)(\kappa-6)(\kappa^2-8\kappa+18)(\kappa^2-4\kappa+6)(\kappa^2-6\kappa+6)^{14}(\kappa^2-6\kappa+12)^{38}.$$

Singular fibers occur at the critical values  $c^2 \in \{0, \pm 1, 7\epsilon_1/16 + i\epsilon_2\sqrt{2}/4, \infty\}$  with  $\epsilon_i = \pm 1$  independently. We have not found any striking feature in their Kodaira types, with are either  $I_1$ ,  $I_2$  or  $I_{16}$ .

## 5.6 The $\lambda \rightarrow 0$ limit

This regime corresponds to  $c \rightarrow 1$ , where the monodromy along the fibers of the flat connection tends to be deterministically equal to identity under the Chern–Simons measure. This implies that  $W(x) \in \mathcal{O}(1)$ , and hence  $\mu_k \rightarrow 0$  since it carries a prefactor of  $\lambda$ . As a result the spectral curve becomes easy to compute:

$$\mathcal{P}_v(x, y)|_{c=1} = C(x) \prod_{i \in I} (y - (-x)^{n_0(w[i])} \zeta_a^{n_1(w[i])}), \quad (5.41)$$

i.e.  $\mathcal{P}_v(x, y)|_{c=1}$  is directly determined by the slope polynomials of  $P_v$ , with no extra data. The description of the orbits leads to the following results.

### $D_{p+2}$ , $p$ even

$$\mathcal{P}_v(x, y)|_{c=1} = (-1)^{p/2+1} (y+1) \cdot ((-x)^{p/2} y^{p/2} + 1) (y^{p/2} - (-x)^{p/2}). \quad (5.42)$$

### $D_{p+2}$ , $p$ odd

$$\begin{aligned} \mathcal{P}_v(x, y)|_{c=1} &= (y+1)^2 \prod_{\ell=0}^{p-1} (y + \zeta_{2p}^\ell x^2) (yx^2 + \zeta_{2p}^{-\ell}) \\ &= (y+1)^2 \cdot (y+x^2)(yx^2+1) \cdot \left( \sum_{k=0}^{p-1} (-1)^k y^{p-1-k} x^k \right) \left( \sum_{k=0}^{p-1} (-1)^k (xy)^k \right). \end{aligned} \quad (5.43)$$

### $E_6$ geometry

$$\mathcal{P}_v(x, y)|_{c=1} = (1+y+y^2) \cdot (x+y)(1+xy) \cdot (y^2-x)(y^2x-1). \quad (5.44)$$

### $E_7$ geometry

$$\mathcal{P}_v(x, y)|_{c=1} = -(1+y)(1+y^2)^2 \cdot (y^2-x^6)(y^2x^6+1) \cdot (x^6+y^3)(x^6y^3-1) \cdot (y^6+x^6)(x^6y^6-1). \quad (5.45)$$

Notice that the symmetry of the orbits implies  $\mathcal{P}(x, y) = 0 \Leftrightarrow \mathcal{P}(\zeta_{12}x, 1/y) = 0$ , explaining how the factors come in pairs.

### $E_8$ geometry

$$\begin{aligned} \mathcal{P}_v(x, y)|_{c=1} &= (y+1)^2 (y^3-1)^3 (y^5-1)^5 \cdot (y^{30}-x^{30})(y^{30}x^{30}-1) \cdot (y^{15}+x^{30})^2 (y^{15}x^{30}+1)^2 \\ &\quad \cdot (y^{10}-x^{30})^2 (y^{10}x^{30}-1)^2 \cdot (y^{15}-x^{60})(y^{15}x^{60}-1) \cdot (y^6-x^{30})(y^6x^{30}-1) \\ &\quad \cdot (y^5+x^{30})(y^5x^{30}+1). \end{aligned} \quad (5.46)$$

## 6 Computations II: the Toda curves

### 6.1 The computation in practice

Let us now construct explicitly the B-model geometries  $\mathcal{C}_{\mathcal{G}_\Gamma}^{\text{Toda}}$  that are relevant for Conjectures 4.1 and 4.2. Recall that we take  $\rho_{\min}$  a minimal representation given in Table 3, and we denote

$d_{\min} = \dim \rho_{\min}$ . The characteristic polynomial of the Lax operator (3.30) for the simple Lie group  $\mathcal{G}$ ,

$$\mathcal{P}_{\mathcal{G}}(Y) \triangleq \det [Y\mathbf{1} - \rho_{\min}(L_{\mathbf{w}}^{\mathcal{G}})] = \sum_{k=0}^{d_{\min}} (-1)^k Y^{d_{\min}-k} \chi_{\Lambda^k \rho}(L_{\mathbf{w}}^{\mathcal{G}}), \quad (6.1)$$

can be regarded as a map

$$\mathcal{P}_{\mathcal{G}} : \mathcal{G}_{\mathbf{w}} \longrightarrow \mathbb{C}[u, Y] \quad (6.2)$$

that factors through a map  $\mathcal{G}_{\mathbf{w}} \longrightarrow \mathfrak{U}_{\mathcal{G}}$  upon evaluation of the antisymmetric characters  $\chi_{\Lambda^k \rho_{\min}} \in \mathbb{Z}[\chi_{\omega_1}, \dots, \chi_{\omega_R}]$  in the representation ring  $\text{Rep}(\mathcal{G})$ . Lifting this to the co-extended affine situation amounts to turning on a spectral parameter as in (3.33). Concretely, we are now looking at the loop space with a map:

$$\tilde{u} : \text{Loop}(\mathfrak{U}_{\mathcal{G}}) \longrightarrow \mathbb{C}[X^{\pm 1}], \quad u_i(X) \triangleq \chi_{\omega_i}(L_{\mathbf{w}}^{\mathcal{G}^{\#}}), \quad (6.3)$$

whose constant term is given by  $[X^0] \tilde{u}_i(X) = u_i$ . The (affine co-extended) Toda spectral curve (3.37) can then be computed in two steps, for each  $(\mathcal{G}, \rho_{\min})$ :

**Step A1'** compute the decomposition of the exterior characters  $\chi_{\Lambda^k \rho_{\min}}$  as polynomials in the fundamental characters  $\chi_{\omega_i}$ ;

**Step A2'** compute from (3.33) the Casimir function  $u_0$  in (3.36) and the dependence of  $\tilde{u}(u_0, \dots, u_R; X) = \chi_{\omega_i}[\rho_{\min}(L_{\mathbf{w}}^{\mathcal{G}^{\#}})]$  on the Hamiltonians  $u_i$  and the spectral parameter  $X$ .

Evaluating the result of Step **A1'** on a generic group element  $g \in \mathcal{G}$  expresses the characteristic polynomial of  $\rho_{\min}(g)$  as

$$\begin{aligned} \det [Y\mathbf{1} - \rho_{\min}(g)] &= \sum_{k=0}^{d_{\min}} (-1)^k Y^{d_{\min}-k} \chi_{\Lambda^k \rho_{\min}}(g), \\ &= \sum_{k=0}^{d_{\min}} Y^k \mathfrak{p}_k^{\mathcal{G}}(g; u_1, \dots, u_R), \end{aligned} \quad (6.4)$$

for some universal<sup>8</sup> polynomials  $\mathfrak{p}_k^{\mathcal{G}} \in \mathbb{Z}[u_1, \dots, u_R]$ , while Step **A2'** amounts to plugging in the expression (3.33) of the Lax matrix and then expand the above in the spectral parameter,

$$\begin{aligned} \mathcal{P}_{\mathcal{G}^{\#}}^{\text{Toda}}(X, Y; u_0, \dots, u_R) &\triangleq \det [Y\mathbf{1} - \rho(L_{\mathbf{w}}^{\mathcal{G}^{\#}})] \\ &= \sum_{k=0}^{d_{\min}} Y^k \mathfrak{p}_k^{\mathcal{G}}[\tilde{u}_1(X), \dots, \tilde{u}_R(X)] \\ &= \sum_{k=0}^{d_{\min}} \sum_{j=-d'_{\min}}^{d'_{\min}} Y^k X^j \mathfrak{p}_{k,j}^{\mathcal{G}}[u_0, u_1, \dots, u_R]. \end{aligned} \quad (6.5)$$

Here  $\mathfrak{p}_{k,j}^{\mathcal{G}}$  denotes the result of the expansion in the spectral parameter  $X$  and

$$d'_{\min} \triangleq \max_{k, \sigma=\pm} \deg_{X^{\sigma}} \mathfrak{p}_k^{\mathcal{G}}. \quad (6.6)$$

---

<sup>8</sup>These polynomials depend implicitly on  $\rho_{\min}$ , but we dropped the subscript  $\rho_{\min}$  for the sake of readability.

The vanishing locus of  $\mathcal{P}_{\mathcal{G}^\#}^{\text{Toda}} \in \mathbb{C}[u_1, \dots, u_R; X^{\pm 1}, Y]$  in  $\mathbb{C}_X^* \times \mathbb{C}_Y^*$  then returns (3.37).

Once this is done, the naive expectation would be that, in light of the discussion of the previous section, all is left to do to prove Point (a) in Conjecture 4.1 is just to find a suitable restriction  $u_i \leftarrow u_i(\lambda)$  of the Toda action variables such that

$$X^{d'_{\min}} \mathcal{P}_{\mathcal{G}^\#}^{\text{Toda}}(X, Y; u(\lambda)) = \mathcal{P}_{\mathcal{G}, v}(X, Y; \lambda), \quad (6.7)$$

where  $\mathcal{P}_{\mathcal{G}, v}(X, Y; \lambda) = 0$  is the spectral curve found in Theorem 5.1, that we call here the “naive LMO spectral curve”. However, this is in general too much to ask.

First off, a rapid inspection of Tables 3 and 4 reveals that the  $Y$ -degrees in (6.7) will disagree in general. But more importantly, the qualitative analysis of the naive LMO spectral curve  $\bar{\mathcal{C}}_v$  given in Section 5.1 reveals that (a) its Galois group  $\mathfrak{W}' = \text{Weyl}(\mathcal{D}') = \text{Weyl}(\mathcal{G}')$  is a subgroup of the Galois group  $\mathfrak{W} = \text{Weyl}(\mathcal{G}) = \text{Weyl}(\mathcal{D})$  of the Toda spectral curve (6.5) with generic parameters  $u$ , and (b) the branchcuts of  $x = X^{1/a} : \bar{\mathcal{C}}_v \rightarrow \mathbb{P}^1$  on the irreducible components of the LMO spectral curve must necessarily be segments obtained from  $x \in [1/\gamma, \gamma]$  by rotations of angle multiple to  $2\pi/a$ , and the branching data of this curve is completely determined by the analysis of orbits of  $\mathfrak{W}'$  in Section 5.3.

This actually suggests a way out of the conundrum: the sought-for subfamily of Toda curves should arise in the sub-locus of the parameter space  $\mathfrak{U}_{\mathcal{G}}$  where the monodromy breaking  $\mathcal{D} \rightarrow \mathcal{D}'$  in Table 2 is enforced. The simplest way to achieve this is to consider an embedding of the subgroup  $\iota : \mathcal{G}' \hookrightarrow \mathcal{G}$ , and the induced embedding  $\iota : \mathcal{T}' \hookrightarrow \mathcal{T}$  of the maximal torus of  $\mathcal{G}'$  into that of  $\mathcal{G}$ . The restriction to  $\iota(\mathcal{T}')$  is cut out by homogeneous linear constraints on the Cartan subalgebra of  $\text{Lie}(\mathcal{G})$ , and its image under the character map  $\chi_\omega$  yields an affine complete intersection  $\tilde{\mathfrak{U}}_{(\mathcal{G}, \mathcal{G}')}$  in  $\mathfrak{U}_{\mathcal{G}}$ . Under the action of  $\iota(\mathcal{G}')$ , the  $\mathcal{G}$ -module  $\rho_{\min}$  decomposes into  $\mathcal{G}'$ -modules:

$$\rho_{\min} = \bigoplus_{j \in J} \rho^{[j]}. \quad (6.8)$$

For  $u \in \mathfrak{U}_{\mathcal{G}}$ , the endomorphism  $\rho_{\min}(L_{\mathbf{w}}^{\mathcal{G}^\#})$  leaves stable the direct sum in (6.8), and thus the characteristic polynomial factors. Requiring that the latter are Laurent polynomials in  $X$  yields an additional constraint, i.e.  $u$  must belong to a subvariety of higher codimension in  $\tilde{\mathfrak{U}}_{(\mathcal{G}, \mathcal{G}')}$ , that we denote  $\mathfrak{U}_{(\mathcal{G}, \mathcal{G}')}$ . In both cases  $(\mathcal{G}, \mathcal{G}') = (E_6, D_4)$  and  $(E_7, E_6)$  we will examine,  $\mathfrak{U}_{(\mathcal{G}, \mathcal{G}')}$  will turn out to be a subvariety of  $\tilde{\mathfrak{U}}_{(\mathcal{G}, \mathcal{G}')}$  ruled along a distinguished direction  $u_{\text{rul}}$ . Summing up:

**Step B'** Consider the Lie group  $\mathcal{G}'$  associated to  $\mathcal{D}'$  as in Table 2, and the decomposition (6.8) as above. Restrict to  $u \in \mathfrak{U}_{(\mathcal{G}, \mathcal{G}')}$  so that

$$\mathcal{P}_{\mathcal{G}^\#}^{\text{Toda}}(X, Y) = \prod_{j \in J} \det[Y \mathbf{1} - \rho^{[j]}(L_{\mathbf{w}}^{\mathcal{G}^\#})] \triangleq \prod_{j \in J} \mathcal{P}_{(\mathcal{G}', \mathcal{G})}^{[j]}(X, Y) \quad (6.9)$$

for Laurent polynomials  $\mathcal{P}_{(\mathcal{G}', \mathcal{G})}^{[j]}(X, Y) \in \mathbb{C}[X^{\pm 1}, Y]$ .

This is the analogue of Step B on the LMO side (Section 5.3), which consists in computing  $\mathcal{P}_{\mathcal{D}, v}(x, y)$  and thus  $\mathcal{P}_{\mathcal{D}}^{\text{LMO}}(X, Y)$  in terms of unknowns  $\mathcal{M} = (\mu_k)_{k \in K_0}$  for a small set  $K_0$ . Let us denote

$\mathcal{P}_{\mathcal{D}}^{\text{LMO}}(X, Y; \mathcal{M}, \lambda)$  this answer. In the following, we will check for  $\mathcal{D} = E_6$  (resp.  $\mathcal{D} = E_7$ ) that with  $K_0 = \{1, 2\}$  (resp.  $K_0 = \{2, 3, 5, 7\}$ ) the equality of polynomials<sup>9</sup> in  $(X, Y)$ :

$$(Y - 1)^{\bullet} \mathcal{P}_{\mathcal{D}}^{\text{LMO}}(X, Y; \mathcal{M}, \lambda) = X^{d'_{\min}} \mathcal{P}_{\mathcal{G}}^{\text{Toda}}(X, Y; u) \quad (6.10)$$

gives an explicit isomorphism

$$\Upsilon : \mathbb{C}_{\mathcal{M}}^{|K_0|} \xrightarrow{\sim} \mathfrak{U}_{\mathcal{G}, \mathcal{G}'}. \quad (6.11)$$

This  $\Upsilon$  is given in (6.37) for  $E_6$  and (6.55) for  $E_7$ . We expect the same property to hold for  $\mathcal{G} = E_8$  (here,  $\mathcal{G}' = E_8$  and  $\mathfrak{U}_{(\mathcal{G}, \mathcal{G}')}$  is just equal to  $\mathfrak{U}_{\mathcal{G}}$ ) but this case was computationally out of reach.

**Step C'** Determine the sublocus of  $u \in \mathfrak{U}_{(\mathcal{G}, \mathcal{G}')}$  such that the Toda spectral curve has the ramification properties that were required for the LMO spectral curve (see Step C, Section 5.3). This should fix  $u$  to live in a 1-dimension variety  $\mathfrak{U}_{\mathcal{G}}^{\text{LMO}}$  parametrized by  $\lambda$ .

In all cases, just by matching the coefficients on the boundary of the Newton polygon, we find that  $u_0 = -1/c^a$ , where we remind that  $c = \exp(\chi_{\text{orb}} \lambda / 2a)$ . Therefore,  $\mathfrak{U}_{\mathcal{G}}^{\text{LMO}}$  is equivalently parametrized by  $(u_i(u_0))_{i=1}^R$ .

By the previous remark, Step C' is strictly equivalent to the determination of  $(\mu_k)_{k \in K_0}$  as functions of  $\lambda$  on the LMO side in Step C. Step A1'-A2'-B' and C' together give a complete derivation of the suitable restriction of the Toda spectral curve, if we assume and verify the qualitative properties used in Step B' and C' that were dictated by the analysis of the matrix model.

## 6.2 $A_{p-1}$ geometries

This is the case of lens spaces  $\mathbb{S}^{\mathbb{Z}/p\mathbb{Z}} = L(p, 1)$  already well-known in the literature, so we will only make a couple of passing remarks here to see how it fits with the discussion above. Steps A1' and A2' were performed in [Nek98, Mar13] and return<sup>10</sup> (3.24), which is in exact agreement with the matrix model curve computed by Halmagyi–Yasnov [HY09] in a general flat background. This proves Point (a) of Conjecture 4.2 for the sphere and disk potential; the rest of Point (a) follows from the solution by the topological recursion method of generalized loop equations [BEO15], which combined with the proof of the remodeling conjecture [EO15] establishes Point (b) as well. The restriction relevant for Step B' and Conjecture 4.1 is simply  $u_0 = e^{-t_B/2} = c^{1/2}$ ,  $u_i = 0$  for  $i \in \llbracket 1, p \rrbracket$ : toric mirror symmetry shows that this amounts to setting to zero the insertion of twisted classes in  $H_{\text{orb}}(Y^{\mathbb{Z}/p\mathbb{Z}})$ , so that the resulting restricted A-model theory is just the untwisted Gromov–Witten theory of the stack  $[\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}/(\mathbb{Z}/p\mathbb{Z})]$ .

## 6.3 $D_{p+2}$ geometries

For  $p + 2 = 4$ , Steps A1'-A2' in this case can be extracted from [KM15] and found to be in agreement with (3.26).

<sup>9</sup>In (6.10), the  $(Y - 1)^{\bullet}$  stands for  $\bullet$  copies of the trivial representation appearing in (6.8) and thus factoring out in the Toda spectral curve..

<sup>10</sup>Choosing the minimal representation  $\rho_{\min}$  to be the anti-fundamental has the sole effect of redefining  $u_k \rightarrow (-1)^p u_{p-k}$ .



### Step A1'

Explicitly, the Dynkin diagram of  $D_{p+2}$  is represented in Figure 4.

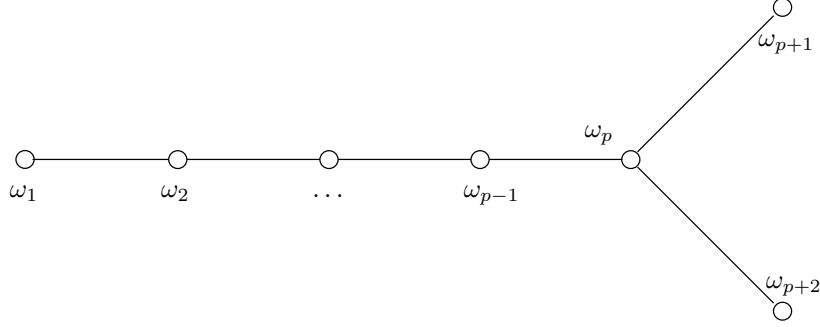


Figure 4: The Dynkin diagram of  $\mathcal{G} = D_{p+2}$ . Nodes in the diagram are labeled by the respective fundamental weights; we have  $\rho_{\min} \triangleq \rho_{\omega_1} = (\mathbf{2p} + \mathbf{4})_{\mathbf{v}}$ ,  $\rho_{\omega_i} = \Lambda^i \rho_{\omega_1}$  for  $i \leq p$ ,  $\rho_{\omega_{p+1}} = \mathbf{2c}^{p+1}$ ,  $\rho_{\omega_{p+2}} = \mathbf{2s}^{p+1}$ .

We write  $\rho_{\min} \triangleq \mathbf{2(p} + \mathbf{2})_{\mathbf{v}}$  for the defining module of  $\text{Spin}(2(p+2))$ . In this specific case, we can slightly bypass Step A1' by parametrizing  $\mathfrak{U}_{\mathcal{G}}$  using the exterior characters  $\epsilon_i = [X^0] \tilde{\epsilon}_i$  with:

$$\tilde{\epsilon}_i \triangleq \chi_{\Lambda^i \rho_{\omega_1}}(L_{\mathbf{w}^{p+2}}^{\#}), \quad \text{for } i \in \llbracket 1, p+2 \rrbracket. \quad (6.12)$$

We have  $\tilde{\epsilon}_i = \tilde{u}_i$  for  $i \leq p$ , and

$$\tilde{\epsilon}_{p+1} = \tilde{u}_{p+1} \tilde{u}_{p+2} - \begin{cases} \sum_{k=0}^{p/2-1} \tilde{u}_{2k+1} & p \text{ even,} \\ \sum_{k=0}^{p/2} \tilde{u}_{2k} & p \text{ odd.} \end{cases} \quad (6.13)$$

$$\tilde{\epsilon}_{p+2} = \tilde{u}_{p+1}^2 + \tilde{u}_{p+2}^2 - 2 \begin{cases} \sum_{k=0}^{p/2} \tilde{u}_{2k} & p \text{ even,} \\ \sum_{k=0}^{p/2} \tilde{u}_{2k+1} & p \text{ odd.} \end{cases} \quad (6.14)$$

as a consequence of the decomposition rules of the tensor products  $\mathcal{S}_{\pm} \otimes \mathcal{S}_{\pm}$  of the chirality  $\pm$  spin representations associated to the fundamental weights  $\omega_{p+1}$  and  $\omega_{p+2}$ .

### Step A2'

The Casimir function  $u_0$  here reads

$$u_0^{-1} = \varkappa_0^{1/2} \varkappa_1 \varkappa_{p+1} \varkappa_{p+2} \prod_{1 < i < p+1} \varkappa_i^2, \quad (6.15)$$

and the Laurent polynomials  $\tilde{\epsilon}_i(X)$  can be computed straightforwardly from (3.33) using Newton identities. We have

$$\begin{aligned} \tilde{\epsilon}_i(X) &= \epsilon_i, & i &\neq p, p+2, p+4, \\ \tilde{\epsilon}_i(X) &= \epsilon_i + u_0(X + 1/X), & i &= p, p+4, \\ \tilde{\epsilon}_{p+2}(X) &= \epsilon_{p+2} - 2u_0(X + 1/X). \end{aligned} \quad (6.16)$$

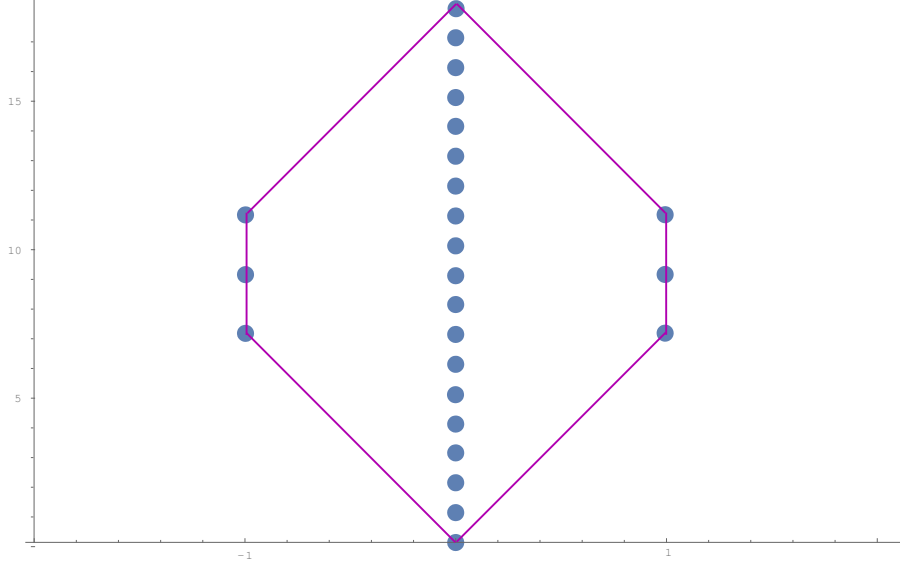


Figure 5: The Newton polygon of the Toda curve for  $\mathcal{G} = D_{p+2}$ , minimal representation  $\rho_{\omega_1} = \mathbf{2}(\mathbf{p} + \mathbf{2})_{\vee}$  for  $p + 2 = 9$ .

The Newton polygon of the resulting plane curve is shown in Figure 5. In terms of exponentiated linear coordinates  $(r_j)_{j=1}^{p+2}$  on the maximal torus  $\mathcal{T}_{D_{p+2}}$ , the resulting curve takes the form

$$\begin{aligned}
 X \mathcal{P}_{D_{p+2}^{\#}}^{\text{Toda}}(X, Y) &= u_0(X^2 + 1)(Y - 1)^2(Y + 1)^2Y^p + \sum_{i=0}^{2(p+2)} (-1)^i \epsilon_i XY^i, \\
 &= u_0(X^2 + 1)(Y - 1)^2(Y + 1)^2Y^p + X \prod_{j=1}^{p+2} (Y - r_j)(Y - r_j^{-1}),
 \end{aligned} \tag{6.17}$$

which is just (3.26) with  $u_0 = e^{-t_B/2}(-1)^{p+1}2^{2p}$ .

### Steps B' and C'

Comparing this Toda curve with the LMO curve (5.37), we find agreement provided  $r_1 = 1$  and:

$$r_j^{\pm 1} = \frac{\tilde{r}_j}{2(\kappa^2 + 1)^2} \pm \sqrt{\frac{\tilde{r}_j^2}{4(\kappa^2 + 1)^4} - 1} \tag{6.18}$$

where  $\tilde{r}_j$  are the  $(p + 1)$  roots of the polynomial  $\mathcal{Q}_p$  given in Appendix C, and:

$$u_0 = (-1)^{p+1} e^{-\lambda/2p}. \tag{6.19}$$

## 6.4 $E_6$ geometry

The Dynkin diagram of  $\mathcal{G} = E_6$  is represented in Figure 6. We write  $\rho_{\min} \triangleq \mathbf{27} = \rho_{\omega_1}$  for the minimal irreducible representation attached to the highest weight  $\omega_1$ ; there is another minimal  $\mathcal{G}$ -module  $\bar{\rho}_{\min} \triangleq \bar{\mathbf{27}} = \rho_{\omega_5}$ , which is complex-conjugate to  $\rho_{\min}$ .

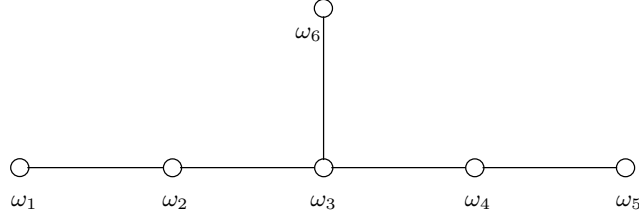


Figure 6: The Dynkin diagram of  $\mathcal{G} = E_6$ . Nodes in the diagram are labeled by the corresponding fundamental weights; we have  $\rho_{\min} \triangleq \rho_{\omega_1} = \mathbf{27} = \overline{\rho_{\omega_5}}$ ,  $\rho_{\omega_2} = \Lambda^2 \rho_{\omega_1} = \mathbf{351} = \overline{\rho_{\omega_4}}$ ,  $\rho_{\omega_3} = \Lambda^3 \rho_{\omega_1} = \mathbf{2925}$ ,  $\rho_{\omega_6} = \text{Adj} = \mathbf{78}$ .

### Step A1'

The fundamental representations of  $E_6$  are antisymmetric powers of  $\rho_{\min}$  and  $\bar{\rho}_{\min}$  with the exception of  $\rho_{\omega_6}$ , which is the 78-dimensional adjoint representation. The antisymmetric characters  $\chi_{\Lambda^k \rho_{\omega_1}}(g) : \mathcal{T}_{E_6} \rightarrow \mathbb{C}$  of an element  $g = e^h$ , which include the fundamental characters  $(\chi_{\omega_i}(g))_{i=1}^5$ , can be computed using the explicit representation of the Chevalley generators in the representations  $\rho_{\min}$  and  $\bar{\rho}_{\min}$  [KM15, HRT01]. On the other hand, the regular character  $\chi_{\omega_6}(g)$  is a trace in the adjoint, which is computed straightforwardly from the root system of  $E_6$ . We must have

$$\chi_{\Lambda^k \rho_{\omega_1}}(g) = \mathfrak{p}_k^{E_6}[\chi_{\omega_1}(g), \dots, \chi_{\omega_6}(g)] \quad (6.20)$$

identically as functions on the Cartan torus. One possible brute-force way to compute  $\mathfrak{p}_k^{E_6}$  is to generate a finite-dimensional vector space of monomials  $\{\chi_{\omega_i}^{n_j}(g)\}_{i,j}$  satisfying a suitable dimensional upper bound on  $\chi_{\omega_i}^{n_j}(1)$ , then evaluate (6.20) at a number of points  $g \in \mathcal{T}$  equal to the dimension of this vector space, and then solve the linear system ensuing from (6.20). The resulting relations

in  $\text{Rep}(\mathcal{G})$  read:

$$\begin{aligned}
\mathfrak{p}_4^{E_6} &= -u_5^2 - u_2u_5 + u_1 + u_4 + u_4u_6, \\
\mathfrak{p}_5^{E_6} &= u_1^2 - 2u_5^2u_1 + 2u_4u_1 + u_4^2 + u_5u_6^2 + u_2 - 2u_3u_5 + u_5 - u_2u_6 - u_5u_6, \\
\mathfrak{p}_6^{E_6} &= -u_5^3 - u_2u_5^2 + u_1u_5 + 2u_4u_5 - 2u_1u_6u_5 + u_4u_6u_5 + u_6^3 + 2u_1u_2 - 2u_3 + u_2u_4 - 3u_3u_6, \\
\mathfrak{p}_7^{E_6} &= 2u_2^2 + u_5u_2 - 2u_5u_6u_2 + u_3u_5^2 + u_1u_6^2 + u_4u_6^2 - 3u_1u_3 - 2u_3u_4 + u_4 - u_5^2u_6 - u_1u_6 + u_4u_6, \\
\mathfrak{p}_8^{E_6} &= u_2u_5^3 - u_1u_6u_5^2 + u_6^2u_5 + u_1u_2u_5 - 2u_3u_5 - 3u_2u_4u_5 + u_3u_6u_5 - 2u_6u_5 + u_5 - u_1^2 - u_2u_6^2 \\
&\quad + u_2u_3 + u_1u_4 + u_1^2u_6 - u_2u_6 + 2u_1u_4u_6, \\
\mathfrak{p}_9^{E_6} &= u_1u_5^4 - u_6u_5^3 + u_2u_5^2 - 4u_1u_4u_5^2 + u_2u_6u_5^2 - u_2^2u_5 - u_1u_6^2u_5 - 4u_1u_5 + 4u_1u_3u_5 + u_4u_5 \\
&\quad + 3u_4u_6u_5 + u_6^3 + u_2^3 + 2u_1u_4^2 + u_1u_2 - 6u_3 + 4u_1^2u_4 - 4u_2u_4 - 3u_3u_6 - 2u_2u_4u_6 + 3, \\
\mathfrak{p}_{10}^{E_6} &= u_5^5 - 5u_4u_5^3 + u_1u_6u_5^3 - u_6^2u_5^2 - u_1u_2u_5^2 + 5u_3u_5^2 - u_5^2 - 2u_1^2u_5 + 5u_4^2u_5 + u_2u_5 + u_2u_3u_5 \\
&\quad + 4u_1u_4u_5 + u_1^2u_6u_5 - 2u_2u_6u_5 - 3u_1u_4u_6u_5 + u_1u_6^2 + 2u_4u_6^2 + 2u_1 + u_1^2u_2 - 5u_1u_3 \\
&\quad + 2u_1u_2u_4 - 5u_3u_4 - u_2^2u_6 - 2u_1u_6 + u_1u_3u_6 - u_4u_6, \\
\mathfrak{p}_{11}^{E_6} &= u_6u_5^4 - u_2u_5^3 + u_1u_3u_5^2 - u_4u_5^2 + 2u_1u_6u_5^2 - 4u_4u_6u_5^2 - 2u_6^2u_5 - u_1u_2u_5 + 3u_3u_5 + 3u_2u_4u_5 \\
&\quad - 2u_1u_2u_6u_5 + 3u_3u_6u_5 - u_6u_5 + u_1u_2^2 + 2u_4^2 + 2u_1^2u_6^2 - 2u_2u_6^2 + 2u_2 - 3u_1^2u_3 + u_2u_3 \\
&\quad + u_2^2u_4 + u_1u_4 - 2u_1u_3u_4 - 2u_1^2u_6 + 2u_4^2u_6 + u_2u_6, \\
\mathfrak{p}_{12}^{E_6} &= 2u_6u_1^3 - u_1^3 + u_5^2u_1^2 - u_4u_1^2 - 2u_2u_5u_1^2 - u_5^2u_6u_1^2 + u_4u_6u_1^2 + 3u_5u_6^2u_1 + 2u_2u_1 - 3u_2u_3u_1 \\
&\quad + u_2u_4u_5u_1 + u_5u_1 - 5u_2u_6u_1 - 2u_5u_6u_1 + u_3^2 + u_3u_5^3 - u_5^3 - 2u_6^3 + 3u_3^2 - u_3 + u_2u_4 \\
&\quad + 3u_2^2u_5 - 3u_3u_4u_5 + 2u_4u_5 + u_5^3u_6 - u_2u_5^2u_6 + 6u_3u_6 - 3u_4u_5u_6, \\
\mathfrak{p}_{13}^{E_6} &= u_1^4 - 2u_5^2u_1^3 + 2u_4u_1^3 + u_4^2u_1^2 - 3u_2u_1^2 - 2u_3u_5u_1^2 - u_5u_1^2 - u_2u_6u_1^2 + 4u_5u_6u_1^2 + 2u_5^3u_1 \\
&\quad + 2u_2u_5^2u_1 - 2u_6^2u_1 + u_3u_1 - 4u_2u_4u_1 + u_2^2u_5u_1 - 4u_4u_5u_1 - u_5^3u_6u_1 + 3u_3u_6u_1 + u_4u_5u_6u_1 \\
&\quad - u_6u_1 + 2u_1 + u_2^2 - 2u_2u_4^2 + u_3u_5^2 + u_2u_4u_5^2 + u_5^2u_6^2 - 3u_3u_4 + 2u_4 + u_2u_5 - u_2u_3u_5 \\
&\quad + u_2^2u_6 - 2u_5^2u_6 - 2u_2u_5u_6,
\end{aligned} \tag{6.21}$$

and  $\mathfrak{p}_{27-k}^{E_6}(u_1, u_2, u_3, u_4, u_5, u_6) = \mathfrak{p}_k^{E_6}(u_5, u_4, u_3, u_2, u_1, u_6)$ . This completes Step **A1'**.

### Step **A2'**

As for the  $D_{p+2}$  case above, the spectral parameter dependence of  $(\tilde{u}_i(X))_{i=1}^5$  can be computed directly from (3.33) using Newton identities. In particular, we obtain

$$\begin{aligned}
\tilde{u}_i(X) &= u_i, & i \neq 3, \\
\tilde{u}_3(X) &= u_3 + u_0(X + 1/X),
\end{aligned} \tag{6.22}$$

in terms of the Casimir function  $u_0^{-1} = \kappa_0^{1/2} \kappa_1 \kappa_2^2 \kappa_3^3 \kappa_4^2 \kappa_5 \kappa_6^2$ . The spectral parameter dependence of  $\tilde{u}_6$  can be computed from the first line of (6.21): the result is

$$\tilde{u}_6(X) = u_6. \tag{6.23}$$

In other words, the  $E_6$ -Toda curve is computed as

$$0 = \mathcal{P}_{E_6^\#}^{\text{Toda}}(X, Y) = \sum_{k=0}^{27} \mathfrak{p}_k^{E_6} [u_1, u_2, u_3 + u_0(X + 1/X), u_4, u_5, u_6] Y^k. \tag{6.24}$$

The resulting Newton polygon is depicted in Figure 7.

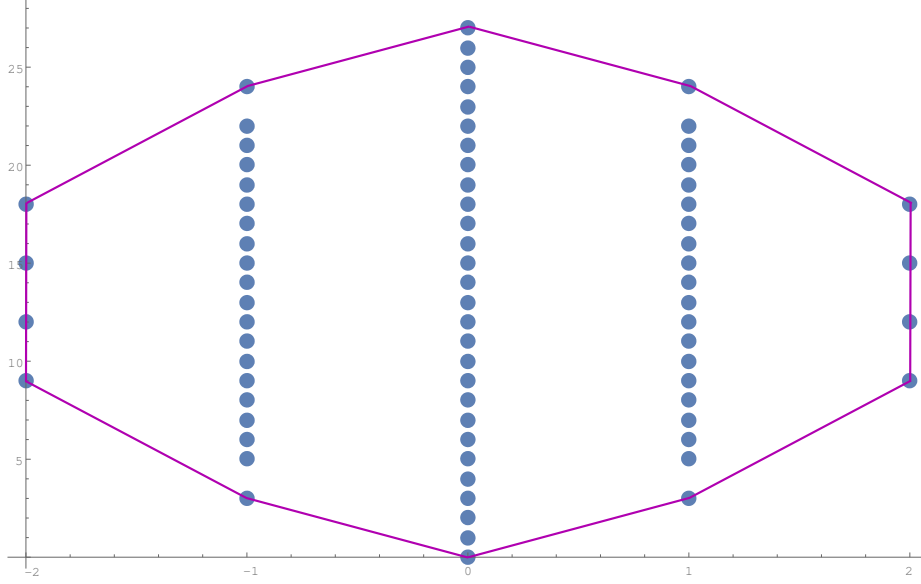


Figure 7: The Newton polygon of the Toda spectral curve for  $\mathcal{G}^\# = E_6^\#$  and  $\rho_{\min} = \mathbf{27}$ .

### Step B'

In this case, we have  $\mathcal{D}' = D_4$  while we started with  $\mathcal{D} = E_6$ . There is an obvious embedding

$$\begin{aligned} \iota : \quad \mathcal{T}_{D_4} &\longrightarrow \mathcal{T}_{E_6} \\ (Q_1, Q_2, Q_3, Q_4) &\longmapsto (1, Q_1, Q_2, Q_3, 1, Q_4) \end{aligned}$$

of the maximal tori, induced by the projection of the weight system of  $E_6$  onto the sublattice of  $\mathbb{Z}^6$  spanned by the unit lattice vectors  $\omega_2, \omega_3, \omega_4$  and  $\omega_6$ . Under this projection, the fundamental highest weight modules of  $E_6$  decompose as  $D_4$ -modules in the following way:

$$\rho_{\omega_1} = \overline{\rho_{\omega_5}} = 3(\mathbf{1}) \oplus \mathbf{8}_s \oplus \mathbf{8}_v \oplus \mathbf{8}_c, \quad (6.25)$$

$$\rho_{\omega_2} = \overline{\rho_{\omega_4}} = 3(\mathbf{1}) \oplus 4(\mathbf{8}_s) \oplus 4(\mathbf{8}_v) \oplus 4(\mathbf{8}_c) \oplus 3(\mathbf{28}) \oplus \mathbf{56}_s \oplus \mathbf{56}_v \oplus \mathbf{56}_c, \quad (6.26)$$

$$\begin{aligned} \rho_{\omega_3} = & 2(\mathbf{1}) \oplus 8(\mathbf{8}_s) \oplus 8(\mathbf{8}_v) \oplus 8(\mathbf{8}_c) \oplus 11(\mathbf{28}) \oplus \mathbf{35}_v \oplus \mathbf{35}_c \oplus \mathbf{35}_s \\ & \oplus 6(\mathbf{56}_s) \oplus 6(\mathbf{56}_v) \oplus 6(\mathbf{56}_c) \oplus 2(\mathbf{160}_s) \oplus 2(\mathbf{160}_v) \oplus 2(\mathbf{160}_c) \oplus \mathbf{350}, \end{aligned} \quad (6.27)$$

$$\rho_{\omega_6} = 2(\mathbf{1}) \oplus 2(\mathbf{8}_s) \oplus 2(\mathbf{8}_v) \oplus 2(\mathbf{8}_c) \oplus \mathbf{28}. \quad (6.28)$$

In particular, with (6.25):

$$\mathcal{P}_{E_6^\#}^{\text{Toda}}(X, Y)|_{u \in \tilde{\mathfrak{U}}_{(E_6, D_4)}} = (Y - 1)^3 \prod_{\bullet = \mathbf{c}, \mathbf{v}, \mathbf{s}} \det[Y\mathbf{1} - \rho_{\bullet}(L_w^{E_6^\#})]. \quad (6.29)$$

The resulting variety  $\tilde{\mathfrak{U}}_{(E_6, D_4)} = (u \circ \iota)(\mathcal{T}')$  is a connected codimension zero submanifold of the intersection of hyperplanes  $u_1 = u_5$ ,  $u_2 = u_4$ , hence it is locally ruled with respect to the  $X$ -dependent Casimir  $u_{\text{rul}} = u_3$ . The degree of the factors in  $Y$ , leaving aside the trivial abelian component  $(Y - 1)^3$ , now reproduces the structure of  $\mathcal{P}_{E_6}^{\text{LMO}}(X, Y)$  as a product over the polynomials associated to the 3 minimal orbits generated by  $v$ ,  $\varepsilon[v]$  and  $\varepsilon^2[v]$  as in (5.14). In  $\mathcal{P}_{E_6}^{\text{LMO}}(X, Y)$ , the 3 factors are polynomials in  $X^{1/3}$  that differ by order 3 rotations  $x \mapsto \zeta_3^j x$ . However, this  $\mathbb{Z}/3\mathbb{Z}$

rotational symmetry is absent in (6.29) for generic values of  $u \in \tilde{\mathfrak{U}}_{\mathcal{G}, \mathcal{G}'}$ ; more importantly, the individual factors appearing in (6.29) are not guaranteed<sup>11</sup> to be polynomials.

This puts two constraints that are solved simultaneously as follows. Denote

$$U_{\bullet} \triangleq \chi_{8_{\bullet}}[L_w^{E_6^{\#}}], \quad U_{\text{adj}} \triangleq \chi_{28}[L_w^{E_6^{\#}}]. \quad (6.31)$$

the evaluation of the  $D_4$ -fundamental characters on the reduced  $L_w^{E_6^{\#}}$  seen for  $u \in \tilde{\mathfrak{U}}_{(E_6, D_4)}$  as a  $D_4$  group element. The following character relations in  $\text{Rep}(D_4)$  are easily deduced from simple tensor multiplication rules:

$$\begin{aligned} \chi_{35_{\bullet}} &= \chi_{8_{\bullet}}^2 - \chi_{28} - 1, \\ \chi_{56_{\bullet}} &= \chi_{8_{\diamond}} \chi_{8_{\star}} - \chi_{8_{\bullet}}, \\ \chi_{160_{\bullet}} &= \chi_{28} \chi_{8_{\bullet}} - \chi_{8_{\diamond}} \chi_{8_{\star}}, \\ \chi_{350} &= \chi_{8_c} \chi_{8_v} \chi_{8_s} - \chi_{8_c}^2 - \chi_{8_v}^2 - \chi_{8_s}^2 + \chi_{28} + 2, \end{aligned} \quad (6.32)$$

where the formulas above should be intended as having the set equality  $\{\bullet, \star, \diamond\} = \{\mathbf{c}, \mathbf{v}, \mathbf{s}\}$ . Also,  $\Lambda^2 8_{\bullet} = 28$ ,  $\Lambda^3 8_{\bullet} = 56_{\bullet}$ ,  $\Lambda^4 8_{\bullet} = 35_{\star} \oplus 35_{\diamond}$ . Then,

$$\begin{aligned} \det_{8_{\bullet}}[Y\mathbf{1} - \mathcal{R}_{8_{\bullet}}(L_w^{E_6^{\#}})] &= Y^8 - U_{\bullet} Y^7 + U_{\text{adj}} Y^6 + (U_{\bullet} - U_{\star} U_{\diamond}) Y^5 + (U_{\diamond}^2 + U_{\star}^2 \\ &\quad - 2U_{\text{adj}} - 2) Y^4 + (U_{\bullet} - U_{\star} U_{\diamond}) Y^3 + U_{\text{adj}} Y^2 - U_{\bullet} Y + 1, \end{aligned} \quad (6.33)$$

and it is immediate to see that restricting to

$$U_{\mathbf{c}} = \tilde{U}^{[2]}(\zeta_3^2 x) + \tilde{U}^{[1]}, \quad U_{\mathbf{s}} = \tilde{U}^{[2]}(\zeta_3 x) + \tilde{U}^{[1]}, \quad U_{\mathbf{v}} = \tilde{U}^{[2]}(x) + \tilde{U}^{[1]}, \quad (6.34)$$

with:

$$\tilde{U}^{[2]}(x) = x + U_0/x, \quad X = x^3, \quad u_0 = U_0^3 \quad (6.35)$$

is necessary and sufficient to attain the required cyclic symmetry with the spectral dependence dictated by (6.22)-(6.23).

### Comparison with the LMO spectral curve

At this stage, we find by direct computation of the left-hand side of (6.10) with (5.14) and the table of coefficients at the beginning of Section 5.5 that the equality:

$$\mathcal{P}_{E_6^{\#}}^{\text{Toda}}(X, Y, u)|_{u \in \mathfrak{U}_{(E_6, D_4)}} = (Y - 1)^3 \mathcal{P}_{E_6}^{\text{LMO}}(X, Y; \mathcal{M}, \lambda), \quad c = e^{\lambda/72}, \quad (6.36)$$

---

<sup>11</sup>This is an instance of the following, general problem: given a family of polynomials  $P \in \mathbb{C}[x, y]$  depending on parameters  $u = (u_i)_{i=1}^R$ , and given an integer  $n \geq 2$ , determine the locus of parameters for which there exists a factorization

$$P(x^n, y; u) = \prod_{j=0}^{n-1} Q(\zeta_n^j x, y; u) \quad (6.30)$$

where  $Q$  is also a polynomial in  $x$  and  $y$ . It was communicated to us by Don Zagier that there is no obvious strategy to solve this problem in general, but one can always try the naive approach consisting in writing down arbitrary coefficients for  $Q$ , expanding (6.30) and solving for the parameters  $(u_i)_{i=1}^R$ . For the example  $(E_6, D_4)$  that we provided, the palindromic symmetry of the factors can be exploited to simplify a bit the derivation.

is realized if and only if:

$$\begin{aligned}\tilde{U}^{[1]} &= -1 - 2c^{-2}\mu_1 \\ U_{\text{adj}} &= 2 + 12c^{-2}\mu_1 - 6c^{-1}\mu_2\end{aligned}\tag{6.37}$$

Eliminating  $\mu_1$  and  $\mu_2$  from these equations, we retrieve exactly the constraints defining  $\mathfrak{U}_{(E_6, D_4)} \subset \mathfrak{U}_{E_6}$ . This is the equivalence between Step **B'** on the Toda side and Step **B** on the LMO side highlighted in Section 6.1. We can then insert in this parametrization the determination of  $\mu_k$ 's in terms of  $\lambda$  (or  $c$ ) performed at the end of Section 5.5. We obtain  $u_0 = -1/c^6$  and the  $(u_i)_{i=1}^6$  as functions of the parameter  $\kappa$  related to  $c$  by (5.40):

$$\begin{aligned}u_1 &= u_5 = \frac{3\kappa(\kappa-4)(\kappa^2-12)(\kappa^2-6\kappa+12)^2}{(\kappa-6)^4(\kappa-2)^4}, \\ u_2 &= u_4 = \frac{3\kappa(\kappa-4)(\kappa^2-6\kappa+12)^2 f_2(\kappa)}{(\kappa-6)^8(\kappa-2)^8}, \\ u_3 &= \frac{f_3(\kappa)}{(\kappa-6)^{12}(\kappa-2)^{12}}, \\ u_6 &= \frac{2f_6(\kappa)}{(\kappa-6)^6(\kappa-2)^6},\end{aligned}\tag{6.38}$$

where  $f_i(\kappa)$  are polynomials given in Appendix D.2.

## 6.5 $E_7$ geometry

The Dynkin diagram of  $\mathcal{G} = E_7$  is represented in Figure 8. We write  $\rho_{\min} \triangleq \mathbf{56} = \rho_{\omega_6}$  for the minimal irreducible representation attached to the highest weight  $\omega_6$ , which is self-dual.

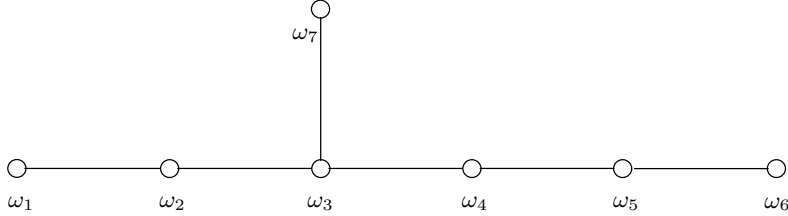


Figure 8: The Dynkin diagram of  $\mathcal{G} = E_7$ . Nodes in the diagram are labeled here by the respective fundamental weights; we have that  $\rho_{\omega_1} = \text{Adj} = \mathbf{133}$ ,  $\rho_{\omega_2} = \mathbf{8645}$ ,  $\rho_{\omega_3} = \mathbf{365750}$ ,  $\rho_{\omega_4} = \mathbf{27664}$ ,  $\rho_{\omega_5} = \mathbf{1539}$ ,  $\rho_{\omega_6} = \rho_{\min} = \mathbf{56}$ ,  $\rho_{\omega_7} = \mathbf{912}$ .

### Step A1'

The computation of  $\mathfrak{p}_k^{E_7}$  can be performed exactly as for the  $E_6$  case. The anti-symmetric characters  $\chi_{\Lambda^k \rho_{\omega_6}}(g)$  can be computed e.g. from the explicit matrix representation of the exponentiated Cartan matrices of  $\rho$  [HRT01] via Newton identities. Also, as before, the fundamental characters  $(\chi_{\omega_k}(g))_{k=1}^6$  are expressed via the characters of suitable tensor powers of  $\rho_{\min} = \rho_{\omega_6}$  and the

character of the adjoint representation  $\mathbf{133} = \text{Adj}$ :

$$\begin{aligned}\chi_{\omega_2} &= \chi_{\Lambda^2 \text{Adj}} - \chi_{\text{Adj}}, \\ \chi_{\omega_3} &= \chi_{\Lambda^4 \omega_6} - \chi_{\Lambda^2 \omega_6}, \\ \chi_{\omega_4} &= \chi_{\Lambda^3 \omega_6} - \chi_{\omega_6}, \\ \chi_{\omega_5} &= \chi_{\Lambda^2 \omega_6} - 1.\end{aligned}\tag{6.39}$$

The remaining fundamental character  $\chi_{\omega_7}$  can be computed from the following relation in  $\text{Rep}(E_7)$

$$\chi_{\omega_7} \chi_{\omega_2} = -(\chi_{\omega_5} + 1) \chi_{\omega_6} - \chi_{\omega_4} + \chi_{\omega_4} \chi_{\omega_1} + \chi_{\omega_6} \chi_{\text{Sym}^2 \omega_6}, \tag{6.40}$$

where the right-hand side can be computed again using Newton identities. Once this is done, the relations  $\mathbf{p}_k^{E_7} = \mathbf{p}_{56-k}^{E_7}$  for  $k \in \llbracket 5, 28 \rrbracket$  in  $\text{Rep}(E_7)$  can be read off by specializing the identity

$$\chi_{\Lambda^k \rho_{\omega_6}}(g) = \mathbf{p}_k^{E_7}[u_1(g), \dots, u_7(g)] \tag{6.41}$$

to a suitably large number of sample points  $g$ , and then solving for the coefficients of  $\mathbf{p}_k^{E_7}$ . We obtain for example that

$$\begin{aligned}\mathbf{p}_5^{E_7} &= -(u_1 - 1) u_4 + (-u_1^2 + u_1 + u_2 + u_5 + 1) u_6 + u_2 u_7, \\ \mathbf{p}_6^{E_7} &= -2u_1^3 + (1 - 2u_5) u_1^2 + (u_6^2 - u_7 u_6 + u_7^2 + 4u_2 - 2u_3 + 2u_5 + 2) u_1 + u_2^2 + u_5^2 - u_3 + 2u_5 \\ &\quad + 2u_2 (u_5 + 1) + u_4 u_6 - u_4 u_7 - u_6 u_7 + 1, \\ \mathbf{p}_7^{E_7} &= u_4 (-u_1^2 + u_1 + u_2 + 2u_5 + 2) + (-u_1^3 + (2u_2 + u_5 + 3) u_1 + 2u_2 - 2u_3 + u_5 + 1) u_6 \\ &\quad + u_7 (-2u_1^2 + (u_2 - 2u_5 + 1) u_1 + u_7^2 + 3u_2 - 3u_3), \\ \mathbf{p}_8^{E_7} &= (u_3 - 2u_5 - u_6 u_7 + 2) u_1^2 + (2u_6^2 + u_4 u_6 - 2u_7 u_6 + u_7^2 + 4u_2 - 4u_3 + 2u_5 - 2u_4 u_7) u_1 \\ &\quad + u_2^2 + 2u_4^2 + u_6^2 + u_5 u_7^2 + u_7^2 - 2u_3 - 3u_3 u_5 + 3u_4 u_6 + u_4 u_7 - u_5 u_6 u_7 + u_6 u_7 \\ &\quad + u_2 (u_6^2 + u_7 u_6 + u_7^2 - 2u_3 + 2u_5) - 2u_1^3, \\ &\dots\end{aligned}\tag{6.42}$$

The expressions up to  $k = 28$  are lengthy and are omitted here, but they are available upon request. This completes Step **A1'**.

### Step **A2'**

As before, the spectral parameter dependence of  $(\tilde{u}_i(X))_{i=3}^6$  can be computed from (3.33) using Newton identities applied to its explicit representation in terms of  $56 \times 56$  matrices. The same holds true for  $(\tilde{u}_i(X))_{i=1,2}$  and explicit 133-dimensional adjoint matrices. Finally,  $\tilde{u}_7(X)$  can be computed from (6.40). We obtain

$$\begin{aligned}\tilde{u}_i(X) &= u_i, & i \neq 3, \\ \tilde{u}_3(X) &= u_3 + u_0(X + 1/X),\end{aligned}\tag{6.43}$$

in terms of the Casimir function  $u_0^{-1} = \varkappa_0^{1/2} \varkappa_1^2 \varkappa_2^2 \varkappa_3^3 \varkappa_4^4 \varkappa_5^3 \varkappa_6^2 \varkappa_7$ . The  $E_7^\#$ -Toda curve is then computed as:

$$0 = \mathcal{P}_{E_7^\#}^{\text{Toda}}(X, Y) = \sum_{k=0}^{56} \mathbf{p}_k^{E_7}[u_1, u_2, u_3 + u_0(X + 1/X), u_4, u_5, u_6, u_7] Y^k. \tag{6.44}$$

The resulting Newton polygon is depicted in Figure 9.



### Step B'

Here we have  $\mathcal{D}' = E_6$  while we started with  $\mathcal{D} = E_7$ . The embedding of the maximal tori

$$\begin{aligned} \iota : \quad \mathcal{T}_{E_6} &\longrightarrow \mathcal{T}_{E_7} \\ (Q_1, Q_2, Q_3, Q_4, Q_5, Q_6) &\longmapsto (Q_1, Q_2, Q_3, Q_4, Q_5, 1, Q_6) \end{aligned}$$

obtained upon projection onto the rank 6 weight sublattice generated by  $\omega_i$ ,  $i \neq 7$  gives rise to the following decomposition of the fundamental weight modules:

$$\rho_{\omega_1} = (\mathbf{1}) \oplus (\mathbf{27}) \oplus (\overline{\mathbf{27}}) \oplus (\mathbf{78}), \quad (6.45)$$

$$\rho_{\omega_2} = (\mathbf{27}) \oplus (\overline{\mathbf{27}}) \oplus 2(\mathbf{78}) \oplus 2(\mathbf{351}) \oplus 2(\overline{\mathbf{351}}) \oplus (\mathbf{650}) \oplus (\mathbf{1728}) \oplus (\overline{\mathbf{1728}}) \oplus (\mathbf{2925}), \quad (6.46)$$

$$\begin{aligned} \rho_{\omega_3} = & (\mathbf{78}) \oplus 3(\mathbf{351}) \oplus 3(\overline{\mathbf{351}}) \oplus 2(\mathbf{650}) \oplus 3(\mathbf{1728}) \oplus (\mathbf{2430}) \oplus 5(\mathbf{2925}) \oplus (\mathbf{5824}) \oplus (\overline{\mathbf{5824}}) \\ & \oplus 3(\mathbf{7371}) \oplus 3(\overline{\mathbf{7371}}) \oplus 2(\mathbf{17550}) \oplus 2(\overline{\mathbf{17550}}) \oplus (\mathbf{34749}) \oplus 2(\mathbf{51975}) \oplus (\mathbf{70070}), \end{aligned} \quad (6.47)$$

$$\begin{aligned} \rho_{\omega_4} = & (\mathbf{27}) \oplus (\overline{\mathbf{27}}) \oplus 2(\mathbf{78}) \oplus 3(\mathbf{351}) \oplus 3(\overline{\mathbf{351}}) \oplus 2(\mathbf{650}) \oplus (\mathbf{1728}) \oplus (\overline{\mathbf{1728}}) \\ & \oplus 2(\mathbf{2925}) \oplus (\mathbf{7371}) \oplus (\overline{\mathbf{7371}}), \end{aligned} \quad (6.48)$$

$$\rho_{\omega_5} = (\mathbf{1}) \oplus 2(\mathbf{27}) \oplus 2(\overline{\mathbf{27}}) \oplus (\mathbf{78}) \oplus (\mathbf{351}) \oplus (\overline{\mathbf{351}}) \oplus (\mathbf{650}), \quad (6.49)$$

$$\rho_{\omega_6} = 2(\mathbf{1}) \oplus (\mathbf{27}) \oplus (\overline{\mathbf{27}}), \quad (6.50)$$

$$\rho_{\omega_7} = (\mathbf{27}) \oplus (\overline{\mathbf{27}}) \oplus 2(\mathbf{78}) \oplus (\mathbf{351}) \oplus (\overline{\mathbf{351}}). \quad (6.51)$$

In particular, with (6.50):

$$\mathcal{P}_{E_7^\#}^{\text{Toda}}(X, Y)|_{u \in \tilde{\mathfrak{U}}_{(E_7, E_6)}} = (Y - 1)^2 \det[Y\mathbf{1} - \rho_{\mathbf{27}}(L_w^{E_7^\#})] \det[Y\mathbf{1} - \rho_{\overline{\mathbf{27}}}(\overline{L_w^{E_7^\#}})]. \quad (6.52)$$

The variety  $\tilde{\mathfrak{U}}_{(E_7, E_6)} = (u \circ \iota)(\mathcal{T}')$ , by (6.45)-(6.51), can be parametrized as the image of the morphism  $u : \mathbb{C}_U^6 \rightarrow \mathfrak{U}_{E_7}$  given by

$$\begin{aligned} u_1 &= U_1 + U_5 + U_6 + 1, \\ u_2 &= U_2 + U_3 + U_4 + U_1 U_5 + U_1 U_6 + U_5 U_6 + U_6 - 1, \\ u_3 &= -U_1^2 + U_3 U_1 + U_4 U_1 - U_1 - U_5^2 + U_2 + 4U_3 + U_2 U_4 \\ &\quad + U_4 + U_2 U_5 + U_3 U_5 - U_5 + U_2 U_6 + U_4 U_6 - 1, \\ u_4 &= U_5 U_2 + 2U_2 + 2U_3 + U_1 U_4 + 2U_4 + 2U_1 U_5 - 2, \\ u_5 &= U_5 U_1 + 2U_1 + U_2 + U_4 + 2U_5, \\ u_6 &= U_1 + U_5 + 2, \\ u_7 &= U_1 + U_2 + U_4 + U_5 + 2U_6. \end{aligned} \quad (6.53)$$

This is not ruled however with respect to  $u_{\text{rul}} = u_3$ , nor can it be expected that the factorization (6.52) give polynomial factors with respect to the spectral parameter  $X$ , let alone have the  $\mathbb{Z}/2\mathbb{Z}$  symmetry of (5.15). The first problem is solved as follows: introduce coordinates  $(\tilde{U}_i)_{i=1}^6$  and  $U_{\text{rul}}$  to parametrize the maximal ruled subvariety  $\tilde{\mathfrak{U}}_{(E_7, E_6)} \times \mathbb{C}_{U_{\text{rul}}}^6$  of  $\mathbb{C}_U^6$  w.r.t. to  $u_{\text{rul}}$ ; we are assuming at this stage the latter to be of dimension higher than zero, with  $U_{\text{rul}}$  a curvilinear coordinate in the distinguished ruling direction. Imposing now that the factorization of  $(Y - 1)^2$  is preserved by shifts along  $u_{\text{rul}}$  has the effect of restricting (6.53) to  $U_5 = U_1 = \tilde{U}_1$ . Furthermore, the requirement that the functions  $u_i : \tilde{\mathfrak{U}}_{(E_7, E_6)} \rightarrow \mathbb{C}$  in (6.53) have vanishing derivative along the distinguished

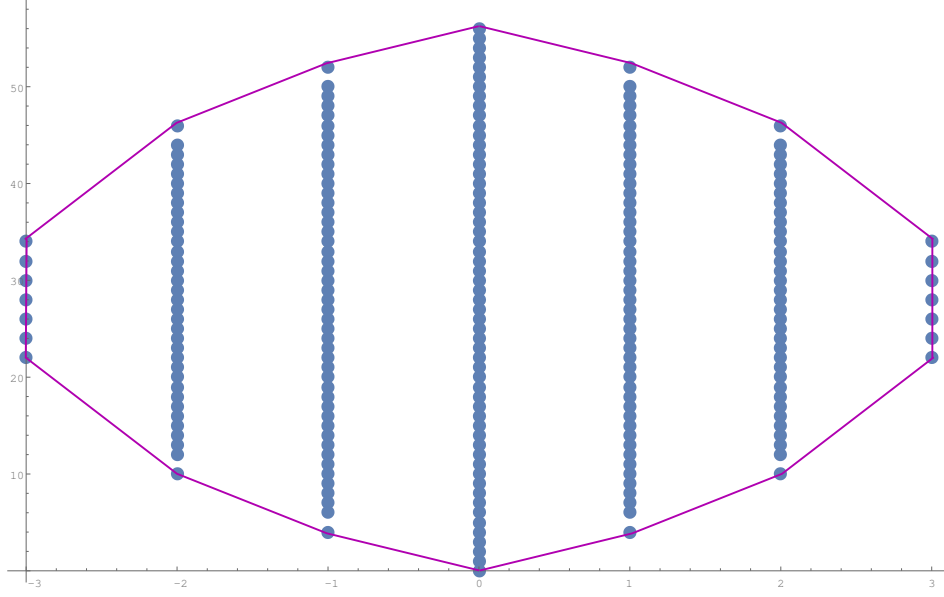


Figure 9: The Newton polygon of the periodic relativistic Toda curve for  $\mathcal{G}^\# = E_7^\#$  and  $\rho_{\min} = \mathbf{56}$ .

direction  $U_{\text{rul}}$  is satisfied upon setting, without loss of generality,  $U_2 = \tilde{U}_2 + U_{\text{rul}}$ ,  $U_4 = \tilde{U}_2 - U_{\text{rul}}$ ,  $U_3 = \tilde{U}_3$ ,  $U_6 = \tilde{U}_4$ . Equating now the third line of (6.53) to the shifted Casimir  $u_3 + X + u_0/X$  as it appears in (6.44) sets  $U_{\text{rul}} = \pm i\sqrt{X + u_0/X}$  with no additional constraints on  $\mathfrak{U}_{E_7, E_6}$ , which thus turns out to have dimension equal to four. Restricting to  $\mathfrak{U}_{E_7, E_6}$  therefore attains the factorization (6.52), which is *a fortiori* polynomial in  $(X, X^{-1})$ . The question of the  $\mathbb{Z}/2\mathbb{Z}$  symmetry in  $X$  is automatically solved by the fact that replacing  $\mathbf{27}$  with  $\overline{\mathbf{27}}$  is tantamount to switching  $U_1 \leftrightarrow U_5$ ,  $U_2 \leftrightarrow U_4$ ; on  $\mathfrak{U}_{(E_7, E_6)}$  this reads  $U_{\text{rul}}^2 \leftrightarrow -U_{\text{rul}}^2$ , which is just  $X \leftrightarrow -X$ . Restricting to  $\mathfrak{U}_{(E_7, E_6)}$  is thus necessary and sufficient to have the factorization in polynomials of  $X$  with the desired  $(\mathbb{Z}/2\mathbb{Z})$ -symmetry.

### Comparison with the LMO spectral curve

By direct computation of the left-hand side of (6.10) with (5.15) and the table of coefficients given in Appendix E.2, we find that the equality:

$$\mathcal{P}_{E_7^\#}^{\text{Toda}}(X, Y, u)|_{u \in \mathfrak{U}_{(E_6, D_4)}} = (Y - 1)^2 \mathcal{P}_{E_7}^{\text{LMO}}(X, Y; \mathcal{M}, \lambda), \quad c = e^{\lambda/72}, \quad (6.54)$$

is realized by  $u_0 = -1/c^{12}$  and the morphism

$$\begin{aligned} \tilde{U}_1 &= -\frac{6\mu_2}{c^3} - 1, \\ \tilde{U}_2 &= \frac{6\mu_5}{c^6} - \frac{12\mu_3}{c^4} + \frac{6\mu_2}{c^3} + 2, \\ \tilde{U}_3 &= \frac{-2c^8 - 24c^5\mu_2 + 24c^4\mu_3 + 36c^2(\mu_2^2 - \mu_5) + 144c\mu_2\mu_3 + 12(2\mu_3^2 + \mu_7)}{c^8}, \\ \tilde{U}_4 &= \frac{12\mu_3}{c^4} + \frac{12\mu_2}{c^3} - 1. \end{aligned} \quad (6.55)$$

## 6.6 $E_8$

The Dynkin diagram of  $\mathcal{G} = E_8$  is represented in Figure 8. We write  $\rho_{\min} \triangleq \text{Adj} = \mathbf{248} = \rho_{\omega_7}$  for the minimal irreducible representation attached to the highest weight  $\omega_7$ : this is the adjoint representation.

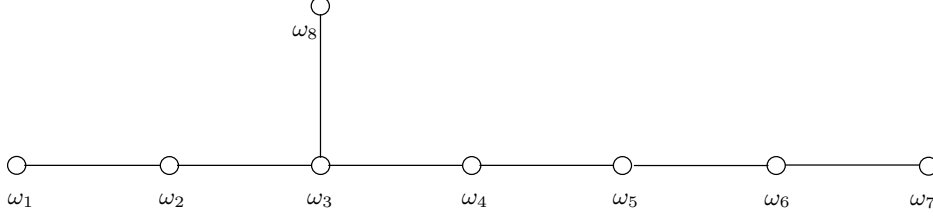


Figure 10: The Dynkin diagram of  $\mathcal{G} = E_8$ . We have  $\rho_{\omega_1} = \mathbf{3875}$ ,  $\rho_{\omega_2} = \mathbf{6696000}$ ,  $\rho_{\omega_3} = \mathbf{6899079264}$ ,  $\rho_{\omega_4} = \mathbf{146325270}$ ,  $\rho_{\omega_5} = \mathbf{2450240}$ ,  $\rho_{\omega_6} = \mathbf{30380}$ ,  $\rho_{\omega_7} = \mathbf{248} = \text{Adj}$ ,  $\rho_{\omega_8} = \mathbf{147250}$ .

### Step A1'

Step **A1'** is the hardest bit here, and the one we could not complete entirely as the size of the linear systems appearing in the calculation of  $\mathfrak{p}_k^{E_8}$  grows uncontrollably all the way up to  $k = 124$ . As a result, we do not have a closed-form expression for  $\mathfrak{p}_k^{E_8}$  but for the first few orders, and in turn we could not find an explicit expression for  $\mathcal{P}_{E_8^\#}$  for arbitrary values of  $u_0, \dots, u_8$ . However, if we are interested in *any given* specific point  $\bar{u} \in \mathfrak{U}_{E_8}$ , and in particular those with integer values for  $\bar{u}_1, \dots, \bar{u}_8$ , the value of  $\mathfrak{p}_k^{E_8}$  at that point can be easily computed in finite time, as follows. Let  $(Q_i)_{i=1}^8$  be exponential coordinates on the maximal torus coming from linear coordinates on  $\text{Lie}(E_8)$ . Then for a given group element  $g$ ,  $u_i = \chi_{\omega_i}(g)$  are Laurent polynomials in the variables  $Q_i$ , and so is  $\chi_{\Lambda^k(\text{Adj})}(g)$  for any  $k$ : the latter in particular can be computed explicitly via Newton identities. For a given  $\bar{u}$ , let  $\bar{Q}$  be any root of the system of algebraic equations  $\chi_{\omega_i}(g) = \bar{u}$ . Plugging  $\bar{Q}$  into the expression of  $\chi_{\Lambda^k(\text{Adj})}(g)$  then returns  $\mathfrak{p}_k^{E_8}|_{u=\bar{u}}$ .

For generic  $\bar{u}$ , it is hopeless to find a manageable expression of  $\bar{Q}$  above that could yield a closed analytic expression for  $\mathfrak{p}_k^{E_8}|_{u=\bar{u}}$ . However, if  $\bar{u} \in \mathbb{Z}^8$ , a sensible thing to do is to find  $\bar{Q}$  numerically to a good accuracy, and then plug the result into the expression of  $\chi_{\Lambda^k(\text{Adj})}(g)$  as a Laurent polynomial in  $\{Q_i\}_{i=1}^8$ : since the latter is on general grounds a polynomial in  $(u_i)_{i=1}^8$  with integer coefficients, it follows that  $\chi_{\Lambda^k(\text{Adj})}(g)|_{Q=\bar{Q}} \in \mathbb{Z}$ . A reliable integer rounding of the numerics gives then a prediction for the *exact* expression of  $\mathfrak{p}_k^{E_8}|_{u=\bar{u}}$ . We will provide an example of this procedure shortly.

### Step A2'

The spectral parameter dependence of  $(\tilde{u}_i(X))_{i=3}^7$  can be computed from (3.33) using Newton identities applied to its explicit representation in terms of  $248 \times 248$  matrices. We obtain

$$\begin{aligned} \tilde{u}_i(X) &= u_i, & i \in \llbracket 4, 7 \rrbracket \\ \tilde{u}_3(X) &= u_3 + u_0(X + 1/X). \end{aligned} \tag{6.56}$$

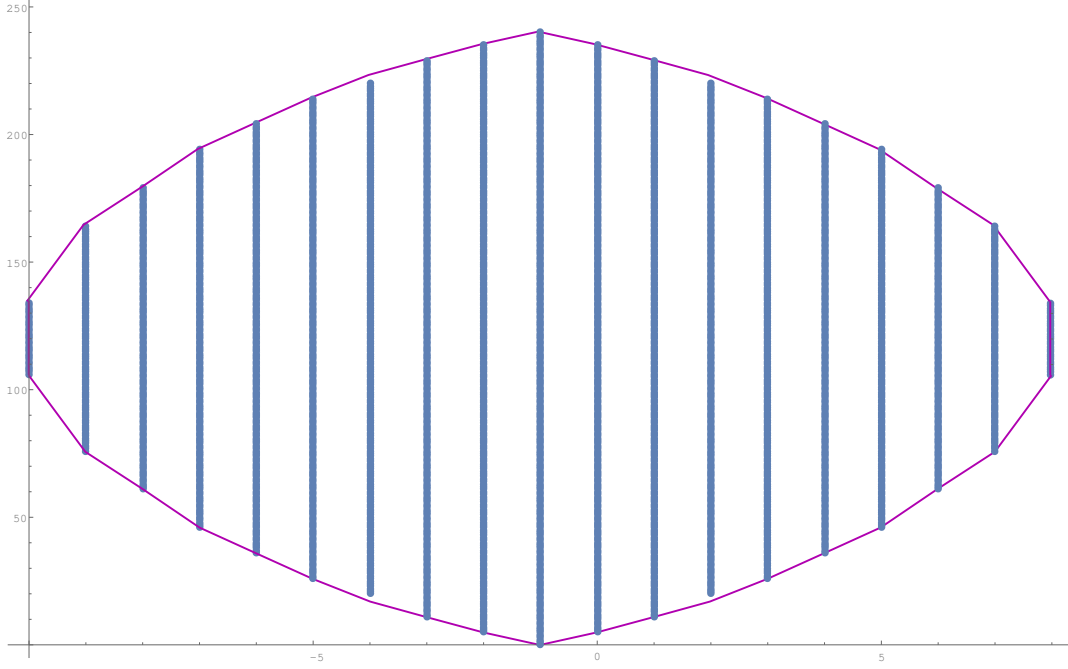


Figure 11: The Newton polygon of the Toda spectral curve for  $\mathcal{G}^\# = E_8^\#$  and  $\rho = 248$ .

in terms of the Casimir function  $u_0^{-1} = \varkappa_0^{1/2} \varkappa_1^2 \varkappa_2^4 \varkappa_3^8 \varkappa_4^5 \varkappa_5^4 \varkappa_6^3 \varkappa_7^2 \varkappa_8^3$ . Furthermore, a quick computation of the character relations  $\mathfrak{p}_k^{E_8}$  for  $k = 6, 7, 8$  reveals that  $\tilde{u}_i(X) = u_i$  for  $i = 1, 2, 8$  as well. The  $E_8^\#$ -Toda curve is then computed as

$$0 = \mathcal{P}_{E_8^\#}^{\text{Toda}}(X, Y; u) = \sum_{k=0}^{248} \mathfrak{p}_k^{E_8} [u_1, u_2, u_3 + u_0(X + 1/X), u_4, u_5, u_6, u_7, u_8] Y^k. \quad (6.57)$$

The polynomials  $\mathfrak{p}_k^{E_8} [\bar{u}_1, \bar{u}_2, \bar{u}_3 + u_0(X + 1/X), \bar{u}_4, \bar{u}_5, \bar{u}_6, \bar{u}_7, \bar{u}_8]$  at  $u_i = \bar{u}_i$ ,  $i \in \llbracket 1, 8 \rrbracket$  can be computed by interpolation of  $\mathfrak{p}_k^{E_8} [\bar{u}_1, \bar{u}_2, \bar{u}_3 + n, \bar{u}_4, \bar{u}_5, \bar{u}_6, \bar{u}_7, \bar{u}_8]$  for  $n \in \llbracket 0, T \rrbracket$  with  $T$  big enough. It turns out that the interpolation stabilizes at  $T = 9$ . The resulting Newton polygon is depicted in Figure 11.

### Step B'

The computational strategy of Step **A1'** above only allows us to compute  $\mathcal{P}_{E_8^\#}^{\text{Toda}}$  at a fixed moduli point  $u = \bar{u}$ , leaving only  $u_0$  unrestricted. This nevertheless leaves some limited space for universal predictions, that are in particular relevant for comparison with  $\mathcal{P}_{E_8}^{\text{LMO}}$ .

Firstly, as  $\mathcal{P}_{E_8^\#}^{\text{Toda}}$  is a characteristic polynomial in the adjoint representation, we automatically have a factor of  $(Y - 1)^8$  pulling out. Factoring out this component leaves us with a degree-240 polynomial  $\tilde{\mathcal{P}}_{E_8^\#}^{\text{Toda}}$  in  $Y$ , as expected from Table 4. Secondly, notice that  $T = 9$  computed in Step **A2'** matches with the fact that  $\deg_X \mathcal{P}_{E_8}^{\text{LMO}} = 18$  in the same table. Thirdly, the palindromic symmetry (5.17) is automatically enforced by (6.57) and the reality of  $\rho_{\min}$ , so that  $\mathfrak{p}_k^{E_8} = \mathfrak{p}_{248-k}^{E_8}$ . Fourthly, in view of all preceding examples, it is natural to assume that the coefficients of the monomials

corresponding to the vertical boundaries should depend on  $u_0$  only. Setting  $u_0 = -1/c^{30}$ , we get:

$$[X^9] \tilde{\mathcal{P}}_{E_8^\#}^{\text{Toda}} = [X^{-9}] \tilde{\mathcal{P}}_{E_8^\#}^{\text{Toda}} = -c^{240} Y^{106} (Y+1)^2 (Y^2+Y+1)^3 (Y^4+Y^3+Y^2+Y+1)^5. \quad (6.58)$$

This is precisely the vertical slope polynomial of the LMO curve given in Appendix F.2.

## 6.7 The LMO slice and the conifold point

The spirit of our calculations so far has been the following: we employed the orbit analysis on the LMO side to enforce the Galois group reduction  $\mathcal{G} \rightarrow \mathcal{G}'$  on  $\mathcal{P}_{\mathcal{G}^\#}^{\text{Toda}}$ , as well as its compatibility with the affine deformation by the spectral parameter  $X$  – i.e. we imposed that the factors of  $\mathcal{P}_{\mathcal{G}^\#}^{\text{Toda}}$  when this reduction are polynomials in  $X$ . It is quite remarkable that such limited piece of data, without any detailed input from the matrix model, allowed us to establish Points (a)-(c) of Conjecture 4.1 – with the sole exception so far of  $\mathcal{G} = E_8$ . However, it would have been desirable to *predict* the restriction  $u_i(\lambda)$  of the Toda action variables relevant for Conjecture 4.1, based on considerations purely within the dual A- and B-model, instead of deriving them *a posteriori* from the comparison with the matrix model curve.

### 6.7.1 Toric case

For  $\mathcal{G} = A_{p-1}$ , a complete interpretation of the LMO restriction can be obtained from the Halmagyi–Yasnov solution of the Chern–Simons matrix model in a generic Chern–Simons vacuum. Let  $[\hat{X}^{\mathbb{Z}/p\mathbb{Z}}] = [\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}/(\mathbb{Z}/p\mathbb{Z})]$  be the  $\mathbb{Z}/p\mathbb{Z}$  fiberwise orbifold of the resolved conifold: its coarse moduli space is the GIT quotient arising from the stability conditions in the maximally singular chamber of the secondary fan of  $Y^{\mathbb{Z}/p\mathbb{Z}} \rightarrow \hat{X}^{\mathbb{Z}/p\mathbb{Z}}$ , which contains [AKMV04] the  $\mathfrak{t} = 0$  point of Chern–Simons theory (see Section 4). The space of marginal deformations of the A-model chiral ring – i.e. the degree-two Chen–Ruan cohomology of  $[\hat{X}^{\mathbb{Z}/p\mathbb{Z}}]$  – is parametrized by linear coordinates  $t = \{t_B, (\tau_{i/p})_{i=1}^{p-1}\}$ : here  $t_B$  is dual to the Kähler class  $c_1(\mathcal{O}_{\mathbb{P}^1}(1))$ , and  $\tau_{i/p}$  are dual to degree zero classes in orbifold cohomology with fermionic age-shift [Zas93] equal to one. A local analysis of the GKZ system around  $t = 0$  then shows that the mirror map in the twisted sector behaves asymptotically as

$$\tau_{i/p} = \mathcal{O}\left(u_k^{i/(kp)}\right), \quad (6.59)$$

for all  $k$  such that  $ik = p$ . Therefore, the LMO restriction  $u_i = 0$ ,  $u_0 = -e^{-t_B/2}$  amounts to switching off the insertion of twisted classes, retaining only the geometric modulus  $t_B = -2\log(c) = -\lambda/p(p+1)$ . This cuts out a 1-dimensional slice of the orbifold chamber of the Kähler moduli space of  $[\hat{X}^{\mathbb{Z}/p\mathbb{Z}}]$ , containing two distinguished boundary points:  $c = 0$  ( $\Re(t_B) = +\infty$ ), corresponding geometrically to the large radius limit point in this orbifold phase (that is, the decompactification  $(\mathbb{Z}/p\mathbb{Z}) \backslash \mathbb{C}^2 \times \mathbb{C} \hookrightarrow [\hat{X}^{\mathbb{Z}/p\mathbb{Z}}]$ ), and the point  $c = 1$  ( $t_B = 0$ ) to  $X_{[0]}^{\mathbb{Z}/p\mathbb{Z}}$ , the  $\mathbb{Z}/p\mathbb{Z}$ -orbifold of the conifold singularity.

### 6.7.2 Non-toric cases

A similar identification does not hold for  $\mathcal{G} = D, E$ , and we are unable to offer a poignant stringy interpretation of the LMO slice here. In this case  $\partial_c u_i(c) \neq 0$  (Figure 12), and as a consequence

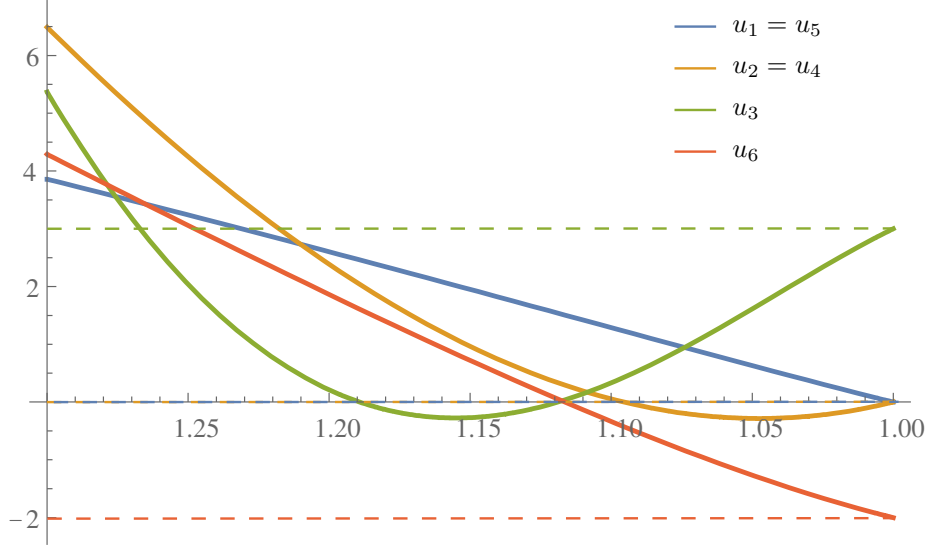


Figure 12: The LMO slice  $u_i(\lambda)$  as a function of  $c = e^{\lambda/72}$  for  $\mathcal{G} = E_6$ . Continuous lines depict the prediction for the Toda Hamiltonians upon restriction to the LMO slice. Dashed lines represent their value on the untwisted slice, where all orbifold moduli have been turned off. It is seen that the two disagree away from the conifold point  $c = 1$ .

the LMO slice does not correspond to the slice where the twisted orbifold moduli are set to zero; it can also readily be seen that the limit  $c \rightarrow 0$  is a different limit point from the (orbifold) large radius point corresponding to  $\Gamma \backslash \mathbb{C}^2 \times \mathbb{C}$ . However, there is at least one special moduli point where we must be able to offer a stringy prediction of the LMO curve with no input from the matrix model: this is the weak 't Hooft limit  $c \rightarrow 1$ , which should correspond to the point in the extended Kähler moduli space corresponding to the  $\Gamma$ -orbifold of the singular conifold,  $X_{[0]}^\Gamma$ , where we have contracted the exceptional curves on each  $\widehat{\Gamma \backslash \mathbb{C}^2} \rightarrow \mathbb{P}^1$  fiber and we further blow-down the base  $\mathbb{P}^1$ . For this point we do have a stringy prediction for the value of  $u_i$ : the Bryan–Graber form of the Crepant Resolution Conjecture [BG09] indeed predicts that the condition of contracting the fibers takes the form, in exponentiated linear coordinates  $Q$  on the Cartan torus  $\mathcal{T}$ ,

$$Q_i = \exp \left( \frac{2\pi i l_i}{|\Gamma|} \right), \quad (6.60)$$

where  $l_i$  is the  $i^{\text{th}}$ -component of the highest root of  $\mathcal{G}$  in the  $\omega$ -basis (equivalently, the dimension of the corresponding irreducible  $\Gamma$ -module). This sets the Toda actions  $u_i$  to the values shown in Table 5. As far as the Kähler modulus of the base  $\mathbb{P}^1$  is concerned, this is related to the Casimir as  $u_0 = -e^{-t_B/2}$ , hence  $u_0 = -1$  is the conifold limit. The conifold B-model curves can then be computed for  $\mathcal{G} = A_{p-1}, D_{p+2}, E_6, E_7$  simply by restriction of the results of Section 6.2-6.5 to  $u_0 = -1$ . Furthermore, since we are sitting at a specific point in the moduli space as per Table 5, we can fully compute the conifold Toda spectral curve for  $\mathcal{G} = E_8$  upon employing the methods of Section 6.6. The results are given in the third column of Table 5 below.

As far as the LMO matrix model is concerned, the small 't Hooft limit is the  $\lambda \rightarrow 0$  limit, for which the large  $N$  spectral curve can be fully determined as we have seen in Section 5.6. Using

$\mathcal{G}$	character values	$\mathcal{P}_{\mathcal{G}^\#}^{\text{Toda}} = 0$
$A_{p-1}$	$u = \epsilon = (0, \dots, 0)$	$X + Y^p X^{-1} = Y^p + 1$
$D_4$	$\epsilon = (0, -2, 0, 2)$	$X + X^{-1} = (Y^2 + Y^{-2})$
$D_{p+2}$	$\epsilon = (0, (-1)^{p+1} 2, 0, (-1)^p, 0, \dots, 0)$	$X + X^{-1} = (Y^p + Y^{-p})$
$E_6$	$u = (0, 0, 3, 0, 0, -2)$	$(Y^3 - 1)^3 (X - Y^3) (XY^3 - 1) (X + Y^6) (XY^6 + 1) = 0$
$E_7$	$u = (-2, 3, -3, 0, 1, 0, 0)$	$(X + 1)^2 (X^2 + 1)^4 (X^4 + Y) (X^6 + Y) (X^{12} + Y) (X^4 Y + 1) (X^6 Y + 1) (X^{12} Y + 1) = 0$
$E_8$	$u = (1, 3, 0, 3, -3, 3, -2, -2)$	$(Y + 1)^2 (Y^2 + Y + 1)^3 (Y^4 + Y^3 + Y^2 + Y + 1)^5 (X + Y^5) (XY^5 + 1) (X - Y^6) (XY^6 - 1) (X - Y^{10})^2 (XY^{10} - 1)^2 (X^2 - Y^{15}) (X + Y^{15})^2 (XY^{15} + 1)^2 (X^2 Y^{15} - 1) (X - Y^{30}) (XY^{30} - 1) (Y - 1)^8 = 0$

Table 5: The values of the B-model moduli at the  $\Gamma$ -orbifold of the conifold point and the corresponding spectral curves. It corresponds to  $u_0 = -1$ . Here,  $(u_i)_{i=1}^R$  are the regular fundamental characters (for  $\mathcal{G} = A, E$ ), and  $(\epsilon_i)_{i=1}^R$  are the antisymmetric characters of the defining representation (for  $\mathcal{G} = A, D$ ).

the definition of  $\mathcal{P}_{\mathcal{D}}^{\text{LMO}}$  in terms of  $\mathcal{P}_v$  given in Section 5.3, we find exact agreement between (5.42)-(5.46) and the Toda spectral curves in Table 5.

## 7 Outlook

We would like to point out a few directions that our findings suggest to explore in relation with existing works.

### The full GOV correspondence

Perhaps the most immediate question is how to extend the LMO/topological strings correspondence of this paper to the full Chern–Simons partition function, so as to give a proof of at least the B-side of the general Conjecture 4.2. For the string side, the relevant family of spectral curves was constructed in Section 6; the only missing ingredient is the full  $E_8$ -curve, whose computation is currently under way [Bri15]. Most of the burden of the comparison is borne by the matrix model side; a good starting point here should be given by the large  $N$  analysis of the matrix integral expression of [Mar04, BT13]. Establishing an explicit solution of the loop equations for this matrix model in terms of the topological recursion applied on the corresponding Toda spectral curve would give a full proof of the B-side of the GOV correspondence.

### Implications for GW theory

The A-side of the correspondence requires substantially more work. While it should be feasible to derive explicit all-genus results e.g. by degeneration techniques [BG08], one should probably work harder to see the Toda spectral setup and the topological recursion emerge. Perhaps the best route to follow here will hinge on deriving the  $S$ - and  $R$ -calibrations of the quantum cohomology of  $Y^\Gamma$ .

emerge from the steepest descent asymptotics of the Toda spectral data, as in [BCR13], and then retrieve the topological recursion from Givental’s  $R$ -action on the associated cohomological field theory [DBOSS14]. This would lead to a proof of the remodeling conjecture beyond the toric case. Along the way it would be interesting to prove a gluing property of ADE invariants analogous to the one enjoyed by the topological vertex. A thorough study of mirror symmetry for the case at hand in the limit  $u_0 \rightarrow \infty$  will appear in [Bri15], where the implications for the Crepant Resolution Conjecture will be explored in detail.

### Implications for gauge theory

The topological recursion method applied to the Toda curves gives us a glimpse of one slice of the  $\Omega$ -background for the associated gauge theory – namely, the one with  $\epsilon_1 = -\epsilon_2$ . It would be very interesting to investigate how the study of the stationary states of their quantized version – which is itself an open problem beyond the  $A$ -case – is related to the twisted superpotential of the gauge theory in the Nekrasov–Shatashvili limit,  $\epsilon_2 = 0$ . Our construction of Section 6 should also embody the solution to the associated  $K$ -theoretic instanton counting problem [GNV09] for the  $A$ - and  $D$ -series; it is natural to imagine, for example, that the extrapolation of the blow-up equation of [GNV09] to exceptional root systems will be solved by the Eynard–Orantin/Nekrasov–Shatashvili free energies of our B-model setup in the respective limits.

### Seifert matrix model and DAHA?

Our results suggest that there should exist observables in the finite  $N$  Seifert matrix model – expressible, moreover, in terms of fiber knot invariants in a spherical Seifert manifold – providing a basis of solutions for the matrix  $q$ -difference equation  $\Psi(qX) = \rho_{\min}(L_w^{\mathcal{G}^\#}(X))\Psi(X)$ . Then, Proposition 1.1 would be the manifestation of this equation in the  $\hbar = \ln q \rightarrow 0$  limit. However, a point of caution must be raised, as here we are considering only the contribution of the trivial flat connection. Therefore, we rather expect  $q$ -difference equations related to affine Toda to be found for observables in the Seifert matrix model with discrete eigenvalues – as opposed to the matrix integral considered here, where the eigenvalues are integrated over the real Cartan subalgebra of  $SU(N)$ . It is likely that the difference between the continuous and the discrete model has no impact on the large  $N$  limit.

Etingof, Gorsky and Losev established in [ELG15, Corollary 1.5] an expression for the colored HOMFLY polynomial of  $(p, q)$  torus knots in  $\mathbb{S}^3$  in terms of characters of the rational double affine Hecke algebra (DAHA) of type  $A_{p-1}$ , which are also related to characters of equivariant  $D$ -modules on the nilpotent cone of  $SL(p)$ . A similar relation for the categorification of the HOMFLY of torus knots was also conjectured in [GORS14], and proved in the uncategorified case. From the present work, we are tempted to think that  $D$  and  $E$  version (instead of  $A_{p-1}$ ) of these results should be expressed in terms of fiber knot invariants in the  $D$  and  $E$  Seifert geometries. Even independently of the knot theory interpretation, establishing that certain observables in Seifert matrix model satisfy exact (for finite  $N$ ) and explicit  $q$ -difference equations produced by DAHA of type  $D$  and  $E$  would be extremely interesting.



### 3d-3d correspondence

From the point of view of the  $3d - 3d$  correspondence [DGG14], Chern–Simons theory on  $M^3$  with simply-laced gauge group  $\exp(\mathfrak{g})$  is dual to a  $3d$  gauge theory  $T_{\mathfrak{g}}[M^3]$  on  $X^3$ : they simultaneously appear in compactification of the  $(2, 0)$   $6d$  SCFT with Lie algebra  $\mathfrak{g}$  on  $M^3 \times X^3$ . The moduli space of classical vacua for  $T_{\mathfrak{su}}[L(p, 1)]$  on  $Y^2 \times \mathbb{S}^1$  is related to the Bethe states in the  $N$  particle sector of the XXZ integrable spin chain on  $p$  sites [GP15]. One can wonder if a direct and thorough relation can be found between the theories  $T_{\mathfrak{sl}}(\mathbb{S}^\Gamma)$  for  $\Gamma$  of type  $D$  and  $E$ , and the  $\mathcal{G}_\Gamma^\#$  classical Toda integrable system, e.g. via spectral dualities in integrable systems.

## A $\pi^1$ and $H_1$ of Seifert spaces

The fundamental group is [Sei80]:

$$\pi_1(M^3) = \left\langle h, c_0, \dots, c_r \mid \begin{array}{l} c_0 h^b = 1, \\ c_m^{a_m} h^{b_m} = [c_m, h] = 1, \quad m \in \llbracket 1, r \rrbracket \\ c_0 \cdots c_r = 1. \end{array} \right\rangle, \quad (\text{A.1})$$

where  $h$  is the generator of a regular fiber, and  $c_1, \dots, c_r$  project to loops around the orbifold points in the base  $\mathbb{S}^2$ . Denoting  $\Sigma$  this orbifold  $\mathbb{S}^2$ , its orbifold fundamental group:

$$\pi_1^{\text{orb}}(\Sigma) = \left\langle c_1, \dots, c_r \mid c_1^{a_1} = \cdots = c_r^{a_r} = \prod_{m=1}^r c_m = 1 \right\rangle \quad (\text{A.2})$$

fits in the exact sequence:

$$\mathbb{Z} \xrightarrow{\iota} \pi_1(M^3) \longrightarrow \pi_1^{\text{orb}}(\Sigma) \longrightarrow 1, \quad (\text{A.3})$$

where  $\iota(\mathbb{Z})$  is the central subgroup generated by  $h$ . One can show that  $\pi_1(M^3)$  is finite iff  $\chi_{\text{orb}} > 0$  and  $\sigma \neq 0$ , which we now assume. Combining the relations in  $\pi_1(M^3)$ , one can show that  $h^{a|\sigma|} = 1$ , but it can happen that the order of  $h$  is smaller than  $a|\sigma|$ . The complete description of the finite fundamental groups appearing here was derived in [Mil57, Ori72] (see also [TZ08] where three misprints of the final list of [Ori72] were corrected), and is summarized below. By Hurewicz theorem, the abelianization of  $\pi_1(M^3)$  gives  $H_1(M^3; \mathbb{Z})$ .

The conditions  $\chi_{\text{orb}} > 0$  and  $\sigma \neq 0$  are satisfied only for  $r = 1, 2$  (lens spaces), and for  $r = 3$  with the orders of exceptional fibers among  $(2, 2, p)$ ,  $(2, 3, 3)$ ,  $(2, 3, 4)$  or  $(2, 3, 5)$ . We introduce the binary polyhedral groups:

- $Q_{4p}$  the binary dihedral group of order  $4p$  (abelianization  $\mathbb{Z}/4\mathbb{Z}$  if  $p$  is odd,  $(\mathbb{Z}/2\mathbb{Z})^2$  if  $p$  is even),
- $P_{24}$  for the symmetry group of the tetrahedron (abelianization  $\mathbb{Z}/3\mathbb{Z}$ ),
- $P_{48}$  for that of the octahedron (abelianization  $\mathbb{Z}/2\mathbb{Z}$ ),
- $P_{120}$  that of the icosahedron (trivial abelianization).

Introduce also the groups:

$$\begin{aligned}
B_{2^k.(2k'+1)} &= \langle x, y \mid x^{2^k} = y^{2k'+1} = xyx^{-1}y = 1 \rangle \\
&\simeq (\mathbb{Z}/(2k'+1)\mathbb{Z}) \rtimes (\mathbb{Z}/2^k\mathbb{Z}), \\
P'_{3^k.8} &= \langle x, y, z \mid x^2 = (xy)^2 = y^2 = zxz^{-1}y^{-1} = zyz^{-1}x^{-1}y^{-1} = z^{3^k} = 1 \rangle \\
&\simeq Q_8 \rtimes (\mathbb{Z}/3^k\mathbb{Z}),
\end{aligned}$$

and for the latter we have  $P'_{3.8} \simeq P_{24}$ . Their abelianizations are:

$$[B_{2^k.(2k'+1)}]_{\text{ab}} = \mathbb{Z}/2^k\mathbb{Z}, \quad [P'_{3^k.8}]_{\text{ab}} = \mathbb{Z}/3^k\mathbb{Z}. \quad (\text{A.4})$$

**(2, 2, p)** –  $\pi_1^{\text{orb}}(\Sigma) = Q_{4p}$ . Denote  $s = p|\sigma|$ . If  $s$  is odd, then  $\pi_1(M^3)$  is the direct product  $(\mathbb{Z}/s\mathbb{Z}) \times Q_{4p}$ . If  $s$  is even, then  $p$  is odd and 4 divides  $s$ ; decompose  $s = 2^{k+1}s'$  with  $s'$  odd; then  $\pi_1(M^3)$  is a non-trivial central extension of  $Q_{4p}$ , namely  $(\mathbb{Z}/s'\mathbb{Z}) \times B_{2^{k+3}.p}$ .

**(2, 3, 3)** –  $\pi_1^{\text{orb}}(\Sigma) = P_{24}$ . Denote  $s = a|\sigma|$ , and decompose  $s = 3^{k-1}s'$  with  $s'$  coprime to 3. If  $k = 1$ , then  $b_2 = b_3 = 1$ ,  $s$  is coprime with 6 and  $\pi_1(M^3)$  is the direct product  $(\mathbb{Z}/s\mathbb{Z}) \times P_{24}$ . If  $k \geq 2$ , then  $(b_2, b_3) = (1, 2)$ ,  $s'$  is coprime to 6, and  $\pi_1(M^3)$  is rather a non-trivial central extension of  $P_{24}$ , namely  $(\mathbb{Z}/s'\mathbb{Z}) \times P'_{8.3^k}$ .

**(2, 3, 4)** –  $\pi_1^{\text{orb}}(\Sigma) = P_{48}$ ,  $a = 12$ , and  $\pi_1(M^3)$  is the direct product  $(\mathbb{Z}/12|\sigma|\mathbb{Z}) \times P_{48}$ .

**(2, 3, 5)** –  $\pi_1^{\text{orb}}(\Sigma) = P_{120}$ ,  $a = 30$ , and  $\pi_1(M^3)$  is the direct product  $(\mathbb{Z}/30|\sigma|\mathbb{Z}) \times P_{120}$ . In particular, for  $b = -1$ ,  $b_1 = b_2 = b_3 = 1$ , we obtain  $30\sigma = 1$ , thus  $\pi_1(M^3) = P_{120}$  and  $H_1(M, \mathbb{Z}) = 0$ . This is the Poincaré sphere, i.e the unique integer homology sphere with finite fundamental group.

In all cases, the order of  $H_1(M^3, \mathbb{Z})$  is  $(\prod_{m=1}^r a_m)|\sigma|$ .

## B Group actions on $\mathbb{S}^3$

It is known that all groups acting smoothly and freely on  $\mathbb{S}^3$  act (up to diffeomorphism) as subgroups of  $\text{SO}(4, \mathbb{R})$ , see e.g. the account in [Zim11]. We now review elementary facts to explain the classification of finite subgroups of  $\text{SO}(4, \mathbb{R})$ . Consider  $\mathbb{S}^3$  as the unit sphere of the quaternions. Thus it forms a Lie group:

$$\mathbb{S}^3 \simeq \text{Sp}(1, \mathbb{H}) \simeq \text{SU}(2, \mathbb{C}) \simeq \text{Spin}(3, \mathbb{R}). \quad (\text{B.1})$$

We have a degree 2 covering of Lie groups:

$$\begin{aligned}
\Phi : \text{SU}(2, \mathbb{C}) &\longrightarrow \text{SO}(3, \mathbb{R}) \\
q &\longmapsto (x \mapsto qxq^{-1})
\end{aligned} \quad (\text{B.2})$$

where in the right-hand side, we consider the linear map restricted to the set of purely imaginary quaternions (it is stable by conjugation since it is the subset of  $\mathbb{H}$  orthogonal to 1). The squared norm of a quaternion is  $\det(q)$ , so the conjugation by  $q$  is an isometry of the 3-space. Since  $\text{SU}(2, \mathbb{C})$  is connected, this isometry always preserve orientation. Since elements of  $\text{SO}(3, \mathbb{R})$  are a fortiori

angle-preserving isomorphism of the sphere, they determine elements of the automorphism group of the Riemann sphere  $\mathbb{S}^2$ , i.e. we have a group homomorphism:

$$\psi : \text{SO}(3, \mathbb{R}) \longrightarrow \text{PSL}(2, \mathbb{C}). \quad (\text{B.3})$$

This is more easily understood starting directly from  $\text{SU}(2, \mathbb{C})$ , since we have the canonical degree 2 covering of Lie groups:

$$\Psi : \text{SU}(2, \mathbb{C}) \rightarrow \text{PSU}(2, \mathbb{C}) \subseteq \text{PSL}(2, \mathbb{C}), \quad (\text{B.4})$$

which factors  $\Psi = \psi \circ \Phi$ . It is not difficult to see that any finite subgroup of  $\text{PSL}(2, \mathbb{C})$  must be conjugated to a finite subgroup of  $\text{PSU}(2, \mathbb{C})$ . There are 3 ways to describe elements in those groups. First, as rotations in  $\mathbb{R}^3$  of angle  $\theta$  around the unit vector  $\vec{x}$ :

$$R(\vec{v}) = \cos \theta \vec{v} + (1 - \cos \theta)(\vec{x} \cdot \vec{v})\vec{x} + \sin \theta \vec{x} \times \vec{v} \quad (\text{B.5})$$

where  $\times$  is the vector product. Second, as  $2 \times 2$  unitary matrices up to a sign:

$$A = e^{\frac{i\pi}{2} \vec{x} \cdot \vec{\sigma}} = \cos(\theta/2) \mathbf{1} + i \sin(\theta/2) \vec{x} \cdot \vec{\sigma} = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}. \quad (\text{B.6})$$

where  $\vec{\sigma}$  is the vector of Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{B.7})$$

Third, as Möbius transformations:

$$z \mapsto \frac{az - \bar{b}}{bz + \bar{a}}. \quad (\text{B.8})$$

These 3 descriptions complement each other.

Then,  $\text{SU}(2, \mathbb{C})$  acts by right and left multiplications on  $\mathbb{S}^3$  (which are isometries on  $\mathbb{S}^3$ ). As a matter of fact, we have a degree 2 covering of Lie groups:

$$\begin{aligned} \Phi_4 : \text{SU}(2, \mathbb{C}) \times \text{SU}(2, \mathbb{C}) &\longrightarrow \text{SO}(4) \\ (q_1, q_2) &\longmapsto (x \mapsto q_1 x q_2^{-1}) \end{aligned} \quad (\text{B.9})$$

which shows that  $\text{Spin}(4, \mathbb{R}) \simeq \mathbb{S}^3 \times \mathbb{S}^3$ . Therefore, the (finite) subgroups of  $\text{SO}(4)$  are of the form  $\Phi_4(G_1 \times G_2)$  where  $G_1, G_2$  are (finite) subgroups of  $\text{SU}(2, \mathbb{C})$ .

By the spin covering (B.2), the finite subgroups of  $\text{SU}(2, \mathbb{C})$  are either cyclic or obtained by adding the matrix  $-\mathbf{1}$  to a finite subgroup of  $\text{PSL}(2, \mathbb{C})$ . The finite subgroups of  $\text{PSL}(2, \mathbb{C})$  are the polyhedral groups. Their extension in  $\text{SU}(2, \mathbb{C})$  are the binary polyhedral groups. Let us give the 3 descriptions of generators for those groups.

- Element  $r_p$ , order  $p$ :

$$r_p = \begin{pmatrix} e^{2i\pi/p} & 0 \\ 0 & e^{-2i\pi/p} \end{pmatrix}. \quad (\text{B.10})$$

$\Phi(r_p)$  is the rotation of angle  $4\pi/p$  around  $\vec{e}_3$  (beware of the factor 2), and  $\Psi(r_p)$  is the Möbius transformation  $z \mapsto e^{4i\pi/p}$ .

- Element  $\iota$ , order 2:

$$\iota = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \quad (\text{B.11})$$

$\Phi(\iota)$  is the symmetry of axis  $\hat{e}_1$ , and  $\Psi(\iota)$  is the inversion  $z \mapsto 1/z$ .

- Element  $j$  of order 3:

$$j = \frac{1}{2} \begin{pmatrix} 1+i & 1-i \\ -1-i & 1-i \end{pmatrix}. \quad (\text{B.12})$$

$\Phi(j)$  is the rotation of angle  $2\pi/3$  around the vector  $\frac{1}{\sqrt{3}}(-\vec{e}_1 + \vec{e}_2 + \vec{e}_3)$ , and  $\Psi(j)$  is the Möbius transformation  $z \mapsto i\frac{z-1}{z+1}$ .

- Element  $\kappa$ , order 2:

$$\kappa = \frac{i}{\sqrt{1+c^2}} \begin{pmatrix} 1 & k \\ k & -1 \end{pmatrix}, \quad k = 2\cos(2\pi/5). \quad (\text{B.13})$$

$\Phi(\kappa)$  is the symmetry of axis  $\frac{1}{\sqrt{1+k^2}}(k\vec{e}_1 + \vec{e}_3)$ , and  $\Psi(\kappa)$  is the Möbius transformation  $z \mapsto \frac{z+k}{kz-1}$ .

Coming back to the list of binary polyhedral groups:  $\mathbb{Z}/p\mathbb{Z}$  is generated by  $r_p$ ;  $\mathbb{Q}_{4p}$  is generated by  $r_{2p}$  and  $\iota$ ;  $P_{24}$  is generated by  $r_4$  and  $j$ ;  $P_{48}$  is generated by  $r_8$  and  $j$ ;  $P_{120}$  is generated by  $r_4$ ,  $j$  and  $\kappa$ .

Classifying the Seifert spaces that are finite quotients of the 3-sphere amounts to classifying the pairs of binary polyhedral groups  $(G_1, G_2)$  such that  $\Phi_4(G_1 \times G_2)$  acts freely on  $\mathbb{S}^3$  in (B.9). Up to isomorphism, this is the list given in Appendix A.

Notice that the action of  $\text{SU}(2)$  by left or right multiplication preserves the symplectic form

$$\omega = i \left( dw_0 \wedge dw_0^* + d\vec{w} \wedge d\vec{w}^* \right) \quad (\text{B.14})$$

on  $\text{Mat}(2, \mathbb{C}) = \{w_0 + i\vec{w} \cdot \vec{\sigma}, \quad w_0, \vec{w} \in \mathbb{C} \times \mathbb{C}^3\}$ . It can be checked by direct computation, using  $(\cos(\theta/2)\mathbf{1} + i\sin(\theta/2)\vec{x} \cdot \vec{\sigma})(w_0 + i\vec{w} \cdot \vec{\sigma}) = w'_0 + i\vec{w}' \cdot \vec{\sigma}$  with

$$\begin{aligned} w'_0 &= \cos(\theta/2)w_0 - \sin(\theta/2)\vec{x} \cdot \vec{w}, \\ \vec{w}' &= \cos(\theta/2)\vec{w} + \sin(\theta/2)w_0\vec{x} - \sin(\theta/2)\vec{x} \times \vec{w}, \end{aligned} \quad (\text{B.15})$$

for a unit vector  $\vec{x}$ , and the properties of the vector product:

$$d(\vec{x} \cdot \vec{w}) \wedge d(\vec{x} \cdot \vec{w}) + (d\vec{x} \times \vec{w}) \wedge (d\vec{x} \times \vec{w}^*) = d\vec{w} \wedge d\vec{w}^*. \quad (\text{B.16})$$

Therefore,  $\Gamma \subset \text{SU}(2)$  acts by symplectomorphisms on the six-fold considered in Section 3.4.

## C $D_{p+2}$ geometry: LMO spectral curve

For all  $p \geq 2$ , it takes the form (5.37):

$$(-1)^{p+1} e^{-\lambda/2p} (X^2 + 1)(Y + 1)^2 + XY (\kappa^2 + 1)^{-(2p+2)} \mathcal{Q}_p[(Y + 1/Y)(\kappa^2 + 1)^2] = 0, \quad (\text{C.1})$$

with

$$\frac{2\kappa^{1+1/p}}{\kappa^2 + 1} = e^{-\lambda/4p^2}. \quad (\text{C.2})$$

$\mathcal{Q}_p$  is a monic polynomial of degree  $p + 1$ . For  $p \leq 5$ , it reads:

$$\begin{aligned} \mathcal{Q}_2(\eta) &= \eta^3 + 2(\kappa^4 + 6\kappa^2 - 3)\eta^2 - 4(\kappa^8 - 4\kappa^6 + 2\kappa^4 + 12\kappa^2 - 3)\eta \\ &\quad - 8(\kappa^2 + 1)^2(\kappa^8 + 14\kappa^4 - 8\kappa^2 + 1), \\ \mathcal{Q}_3(\eta) &= \eta^4 + 8(2\kappa^2 - 1)\eta^3 - 8(\kappa^8 - 4\kappa^4 + 12\kappa^2 - 3)\eta^2 - 32(2\kappa^{10} + 3\kappa^8 + 8\kappa^6 + 4\kappa^4 - 6\kappa^2 + 1)\eta \\ &\quad + 16(\kappa^2 + 1)^2(\kappa^2 - 1)(\kappa^4 - 4\kappa^2 + 1)(\kappa^6 + 3\kappa^4 + 5\kappa^2 - 1), \\ \mathcal{Q}_4(\eta) &= \eta^5 - 2(\kappa^4 - 10\kappa^2 + 5)\eta^4 - 8(\kappa^8 + 4\kappa^6 - 12\kappa^4 + 20\kappa^2 - 5)\eta^3 \\ &\quad + 16(\kappa^{12} - 6\kappa^{10} - 11\kappa^8 - 8\kappa^6 - 33\kappa^4 + 30\kappa^2 - 5)\eta^2 \\ &\quad + 16(\kappa^{16} + 8\kappa^{14} - 8\kappa^{12} - 24\kappa^{10} + 64\kappa^8 - 30\kappa^6 + 56\kappa^4 - 40\kappa^2 + 5)\eta \\ &\quad - 32(\kappa^2 + 1)^2(\kappa^{16} - 4\kappa^{14} - 4\kappa^{12} - 4\kappa^{10} - 10\kappa^8 - 44\kappa^6 + 44\kappa^4 - 12\kappa^2 + 1), \\ \mathcal{Q}_5(\eta) &= \eta^6 - 4(\kappa^4 - 6\kappa^2 + 3)\eta^5 - 4(\kappa^8 + 20\kappa^6 - 46\kappa^4 + 60\kappa^2 - 15)\eta^4 \\ &\quad + 32(\kappa^{12} - 2\kappa^{10} - 15\kappa^8 + 12\kappa^6 - 41\kappa^4 + 30\kappa^2 - 5)\eta^3 \\ &\quad - 16(\kappa^{16} - 24\kappa^{14} + 4\kappa^{12} + 40\kappa^{10} + 6\kappa^8 + 24\kappa^6 - 236\kappa^4 + 120\kappa^2 - 15)\eta^2 \\ &\quad - 64(\kappa^{20} + 2\kappa^{18} - 17\kappa^{16} - 24\kappa^{14} - 30\kappa^{12} - 52\kappa^{10} - 98\kappa^8 + 8\kappa^6 + 77\kappa^4 - 30\kappa^2 + 3)\eta \\ &\quad + 64(\kappa^2 + 1)^2(\kappa^2 - 1)(\kappa^8 - 2\kappa^6 + 2\kappa^4 - 6\kappa^2 + 1)(\kappa^{10} - 3\kappa^8 - 12\kappa^6 - 8\kappa^4 + 7\kappa^2 - 1). \end{aligned}$$

At  $c = 0$ , i.e.  $\kappa = 0$ , we always have  $\mathcal{Q}_p(\eta)|_{\kappa=0} = (\eta - 2)^{p+1}$ . At  $c = 1$ , i.e.  $\kappa = 1$ , the roots of those polynomials are:

$$\begin{aligned} p = 2 & \quad -2, \pm\sqrt{2} \\ p = 3 & \quad 0, 2, \pm\sqrt{3} \\ p = 4 & \quad -2, \epsilon_1\sqrt{2 + \epsilon_2\sqrt{2}} \\ p = 5 & \quad 0, -2, \frac{\epsilon_1}{2}\sqrt{10 + 2\epsilon_2\sqrt{2}} \end{aligned}$$

where  $\epsilon_i = \pm 1$ .

## D $E_6$ geometry

### D.1 LMO side: discriminant of $\mathcal{R}$ in (5.39)

The discriminant has two factors:

$$\begin{aligned} \Delta_1 &= -32 + 27c^2 + 68c^4 + 160\mu_1^2 + 32\mu_1 - 32\mu_1^3 - 144\mu_2^2 - 256\mu_1^4 - 128\mu_1^5 \\ &\quad + 31c^{10} - 36c^8 - 58c^6 + 408c^2\mu_1 - 408c\mu_2 - 864c\mu_2^3 - 144\mu_2^2\mu_1^2 - 288\mu_2^2\mu_1 \\ &\quad - 1296c^2\mu_2^2 - 432c^4\mu_1 + 372c^3\mu_2 - 648c^2\mu_1^2 - 396c^7\mu_2 + 1832c^6\mu_1^2 \\ &\quad + 432c^8\mu_1 + 3104c^4\mu_1^3 + 1404c^4\mu_2^2 + 1776c^2\mu_1^4 - 1440c^2\mu_1^3 + 648c\mu_1\mu_2 \\ &\quad - 3240c^5\mu_1\mu_2 - 5136c^3\mu_2\mu_1^2 + 3888c^2\mu_2^2\mu_1 + 576c\mu_1^2\mu_2 + 2688c^3\mu_1\mu_2 \\ &\quad - 480c\mu_1^3\mu_2 + 432c^5\mu_2 - 440c^6\mu_1 - 1392c^4\mu_1^2, \\ \Delta_2 &= 32 + 27c^2 - 68c^4 - 160\mu_1^2 + 32\mu_1 - 32\mu_1^3 - 144\mu_2^2 + 256\mu_1^4 - 128\mu_1^5 + 31c^{10} \\ &\quad + 36c^8 - 58c^6 - 408c^2\mu_1 + 408c\mu_2 - 864c\mu_2^3 - 144\mu_2^2\mu_1^2 + 288\mu_2^2\mu_1 \\ &\quad + 1296c^2\mu_2^2 - 432c^4\mu_1 + 372c^3\mu_2 - 648c^2\mu_1^2 - 396c^7\mu_2 + 1832c^6\mu_1^2 + 432c^8\mu_1 \\ &\quad + 3104c^4\mu_1^3 + 1404c^4\mu_2^2 + 1776c^2\mu_1^4 + 1440c^2\mu_1^3 + 648c\mu_1\mu_2 - 3240c^5\mu_1\mu_2 \\ &\quad - 5136c^3\mu_2\mu_1^2 + 3888c^2\mu_2^2\mu_1 - 576c\mu_1^2\mu_2 - 2688c^3\mu_1\mu_2 - 480c\mu_1^3\mu_2 \\ &\quad - 432c^5\mu_2 + 440c^6\mu_1 + 1392c^4\mu_1^2. \end{aligned}$$

## D.2 Toda side: the polynomials $f_i(\kappa)$ in (6.38)

$$\begin{aligned}
f_2(\kappa) &= 248832 - 912384\kappa + 1119744\kappa^2 - 617472\kappa^3 + 115584\kappa^4 + 43776\kappa^5 - 32096\kappa^6 + 8704\kappa^7 \\
&\quad - 1260\kappa^8 + 96\kappa^9 - 3\kappa^{10}, \\
f_3(\kappa) &= 26748301344768 - 231818611654656\kappa + 922816396394496\kappa^2 - 2265845304655872\kappa^3 \\
&\quad + 3881228059017216\kappa^4 - 4961293921419264\kappa^5 + 4932729950699520\kappa^6 \\
&\quad - 3918448994549760\kappa^7 + 2531494971703296\kappa^8 - 1345368215715840\kappa^9 \\
&\quad + 592090245808128\kappa^{10} - 216319699795968\kappa^{11} + 65493454344192\kappa^{12} \\
&\quad - 16315792478208\kappa^{13} + 3293224915968\kappa^{14} - 521639046144\kappa^{15} + 60092669952\kappa^{16} \\
&\quad - 3803240448\kappa^{17} - 196007424\kappa^{18} + 88858368\kappa^{19} - 12725856\kappa^{20} + 1122176\kappa^{21} \\
&\quad - 64176\kappa^{22} + 2208\kappa^{23} - 35\kappa^{24}. \\
f_6(\kappa) &= -2985984 + 13436928\kappa - 24758784\kappa^2 + 26085888\kappa^3 - 17843328\kappa^4 + 8404992\kappa^5 \\
&\quad - 2802048\kappa^6 + 667008\kappa^7 - 112752\kappa^8 + 13248\kappa^9 - 1032\kappa^{10} + 48\kappa^{11} - \kappa^{12}.
\end{aligned}$$

## E $E_7$ geometry

### E.1 LMO side: minimal orbit

The orbit consists of 27 12-dimensional vectors with entries  $-1, 0, 1$ . If  $w$  is in the orbit, so does  $-\varepsilon(w) = (-w_{j+1 \bmod a})_j$ . Below we give an element of  $\{\pm w, -\varepsilon(w), \varepsilon^2(w), \dots\}$  that has  $n_0(w) = \sum_{k=0}^{11} w_k \geq 0$ , and indicate the size  $l$  of this sub-orbit. We encode the vectors in  $\underline{w}(t) = \sum_{k=0}^{11} w_k t^k$ .

$n_0$	$l$	$\underline{w}(t)$
$\pm 3$	4	$1 + t^4 + t^8$
$\pm 2$	6	$t - t^2 + t^3 + t^7 - t^8 + t^9$
$\pm 1$	12	$1 + t^6 - t^7 + t^9 - t^{11}$
0	4	$-1 + t^3 - t^4 + t^7 - t^8 + t^{11}$
0	1	$\sum_{k=0}^{11} (-1)^k t^k$

### E.2 LMO spectral curve in terms of $\mu_k$ 's

We find  $\mathcal{P}_v(x, y) = \sum_{j=0}^6 \sum_{k=0}^{27} \Pi_{j,k} x^{6j} y^k$  with the symmetries  $\Pi_{j,k} = (-1)^{j+1} \Pi_{6-j,k} = (-1)^j \Pi_{j,27-k}$ . All non-zero coefficients are deduced by symmetry from the following list, and depend on the 4 parameters  $\mu_2, \mu_3, \mu_5$  and  $\mu_7$  which are unknown functions of  $c$ :

$$\begin{aligned}
\Pi_{0,11} &= \Pi_{0,12} = -c^{-18} \\
\Pi_{0,13} &= -2c^{-18} \\
\Pi_{1,5} &= \Pi_{1,6} = -c^{-12} \\
\Pi_{1,7} &= -2c^{-12} \\
\Pi_{1,8} &= -3c^{-15}(c^3 + 6\mu_2) \\
\Pi_{1,9} &= -c^{-16}(c^4 + 18c\mu_2 + 24\mu_3) \\
\Pi_{1,10} &= -2c^{-16}(c^4 + 15c\mu_2 + 6\mu_3) \\
\Pi_{1,11} &= -6c^{-18}(5c^3\mu_2 + 6c^2\mu_3 - \mu_5) \\
\Pi_{1,12} &= c^{-18}(c^6 - 12c^3\mu_2 - 36\mu_2^2 - 36c^2\mu_3 + 18\mu_5) \\
\Pi_{1,13} &= -c^{-18}(c^6 + 12c^3\mu_2 + 36\mu_2^2 - 12c^2\mu_3 + 12\mu_5) \\
\Pi_{2,2} &= -c^{-6} \\
\Pi_{2,3} &= 0 \\
\Pi_{2,4} &= c^{-10}(-c^4 + 6c\mu_2 + 12\mu_3) \\
\Pi_{2,5} &= -12c^{-12}(c^3\mu_2 - c^2\mu_3 + \mu_5) \\
\Pi_{2,6} &= 2c^{-13}(c^3 + 6\mu_2)(c^4 - 3c\mu_2 - 6\mu_3) \\
\Pi_{2,7} &= -6c^{-14}(c^5\mu_2 - 6c^4\mu_3 - 24c\mu_2\mu_3 + 16\mu_3^2 + c^2(-12\mu_2^2 + 8\mu_5) - 4\mu_7) \\
\Pi_{2,8} &= 3c^{-15}(c^9 + 72\mu_2^3 - 20c^5\mu_3 + 12c^3(\mu_2^2 + \mu_5) + c(40\mu_3^2 - 4\mu_7)) \\
\Pi_{2,9} &= c^{-16}(c^3 + 6\mu_2)(c^7 + 12c^4\mu_2 - 24c^3\mu_3 + 72\mu_2\mu_3 + 36c(-2\mu_2^2 + \mu_5)) \\
\Pi_{2,10} &= -6c^{-17}(3c^8\mu_2 + 4c^7\mu_3 - 24c^4\mu_2\mu_3 + 12c^5(2\mu_2^2 - \mu_5) + 24c\mu_3(3\mu_2^2 - \mu_5) \\
&\quad + 36c^2(\mu_2^3 + 2\mu_2\mu_5) + 8c^3(2\mu_3^2 + \mu_7) - 12\mu_2(2\mu_3^2 + \mu_7)) \\
\Pi_{2,11} &= c^{-18}(3 + c^{12} + 24c^9\mu_2 - 60c^8\mu_3 - 432c^5\mu_2\mu_3 + 36\mu_5^2 - 432c^2\mu_3(2\mu_2^2 + \mu_5) \\
&\quad + c^6(-36\mu_2^2 + 48\mu_5) + 72c^3(9\mu_2^3 + 2\mu_2\mu_5) + 12c^4(34\mu_3^2 - \mu_7)) \\
&\quad - 144c\mu_2(2\mu_3^2 + \mu_7)) \\
\Pi_{2,12} &= -3c^{-18}(-1 + c^{12} + 8c^9\mu_2 - 144c^3\mu_2^3 - 24c^8\mu_3 - 48c^5\mu_2\mu_3 + 144c^4\mu_3^2 \\
&\quad + 36\mu_5^2 + 12c^6(2\mu_2^2 + \mu_5) - 144c^2\mu_3(2\mu_2^2 + \mu_5)) \\
\Pi_{2,13} &= 2c^{-18}(3 - c^{12} + 3c^9\mu_2 + 24c^8\mu_3 + 72c^5\mu_2\mu_3 + 6c^6(15\mu_2^2 - 8\mu_5) \\
&\quad - 72c^2\mu_3\mu_5 + 36\mu_5^2 - 36c^3(9\mu_2^3 - \mu_2\mu_5) + 24c^4(2\mu_3^2 + \mu_7) \\
&\quad + 36c\mu_2(2\mu_3^2 + \mu_7)) \\
\Pi_{3,0} &= 1 \\
\Pi_{3,1} &= 1 + 6c^{-3}\mu_2 \\
\Pi_{3,2} &= 2 + 6c^{-3}\mu_2 - 12c^{-4}\mu_3 + 6c^{-6}\mu_5 \\
\Pi_{3,3} &= 2c^{-8}(c^8 + 12c^5\mu_2 - 12c^4\mu_3 - 72c\mu_2\mu_3 - 18c^2(\mu_2^2 - \mu_5) - 6(2\mu_3^2 + \mu_7)) \\
\Pi_{3,4} &= 6c^{-10}(4c^7\mu_2 + 2c^6\mu_3 - 24c^3\mu_2\mu_3 - 24c^2\mu_3^2 + 18c\mu_2\mu_5 + 12\mu_3\mu_5 \\
&\quad + c^4(12\mu_2^2 + \mu_5)) \\
\Pi_{3,5} &= 2c^{-12}(1 + 15c^9\mu_2 - 6c^8\mu_3 - 288c^5\mu_2\mu_3 + 24c^6\mu_5 + 252c^3\mu_2\mu_5 \\
&\quad + 108c^2\mu_3\mu_5 - 18\mu_5^2 + 72c\mu_2(4\mu_3^2 - \mu_7) - 12c^4(8\mu_3^2 + \mu_7)) \\
\Pi_{3,6} &= c^{-13}(-3c^{13} + 30c^{10}\mu_2 + 84c^9\mu_3 - 864c^6\mu_2\mu_3 - 216c^3\mu_3(6\mu_2^2 - 5\mu_5) \\
&\quad - 432\mu_2\mu_3\mu_5 - 36c^7(\mu_2^2 + \mu_5) - 36c^4(24\mu_2^3 - 31\mu_2\mu_5) \\
&\quad - 432c^2\mu_2\mu_7 + 12c^5(-82\mu_3^2 + \mu_7) \\
&\quad + c(2 + 864\mu_3^3 - 648\mu_2^2\mu_5 + 36\mu_5^2 - 432\mu_3\mu_7)) \\
\Pi_{3,7} &= -c^{-14}(3c^{14} + 12c^{11}\mu_2 - 72c^{10}\mu_3 + 360c^7\mu_2\mu_3 + 864c^6\mu_3^2 \\
&\quad + 72c^4\mu_3(18\mu_2^2 - 19\mu_5) + 864c\mu_2\mu_3(6\mu_2^2 + \mu_5) + 12c^8(6\mu_2^2 + 5\mu_5) \\
&\quad + 36c^5(6\mu_2^3 - 7\mu_2\mu_5) - 216c^3\mu_2(6\mu_3^2 - \mu_7) \\
&\quad + 4c^2(-1 + 324\mu_2^4 - 288\mu_3^3 + 126\mu_5^2 + 72\mu_3\mu_7) \\
&\quad + 144(\mu_5(4\mu_3^2 - \mu_7) + 3\mu_2^2(2\mu_3^2 + \mu_7)))
\end{aligned}$$

$$\begin{aligned}
\Pi_{3,8} &= -3c^{-15}(c^{15} + 4c^{12}\mu_2 - 288c^6\mu_2^3 - 20c^{11}\mu_3 - 168c^5\mu_3\mu_5 \\
&\quad + 288c^2\mu_2\mu_3(9\mu_2^2 + 2\mu_5) + c^9(-36\mu_2^2 + 38\mu_5) + 4c^7(26\mu_3^2 - 5\mu_7) \\
&\quad - 48c^4\mu_2(8\mu_3^2 + \mu_7) + 12\mu_2(-1 + 48\mu_3^3 + 36\mu_2^2\mu_5 - 18\mu_5^2 + 24\mu_3\mu_7) \\
&\quad + c^3(-2 + 432\mu_2^4 - 384\mu_3^3 - 648\mu_2^2\mu_5 + 36\mu_5^2 + 96\mu_3\mu_7) \\
&\quad + 24c(\mu_5(10\mu_3^2 - \mu_7) + 12\mu_2^2(14\mu_3^2 + \mu_7))) \\
\Pi_{3,9} &= c^{-16}(-3c^{16} - 78c^{13}\mu_2 + 36c^{12}\mu_3 + 1296c^9\mu_2\mu_3 - 432c^3\mu_2\mu_3(24\mu_2^2 - 19\mu_5) \\
&\quad + 18c^{10}(2\mu_2^2 - 5\mu_5) - 144c^6\mu_3(3\mu_2^2 - 5\mu_5) + 108c^7(4\mu_2^3 - 9\mu_2\mu_5) \\
&\quad - 432c^5\mu_2(10\mu_3^2 - \mu_7) + 36c^8(-2\mu_3^2 + \mu_7) \\
&\quad + 36c\mu_2(1 + 216\mu_2^4 - 192\mu_3^3 - 216\mu_2^2\mu_5 + 30\mu_5^2 - 96\mu_3\mu_7) \\
&\quad - 2c^4(-1 + 2592\mu_2^4 - 720\mu_3^3 - 3456\mu_2^2\mu_5 + 594\mu_5^2 + 72\mu_3\mu_7) \\
&\quad - 48(12\mu_3^4 + \mu_3(-1 + 54\mu_2^2\mu_5 - 18\mu_5^2) + 12\mu_3^2\mu_7 + 3\mu_7^2) \\
&\quad - 864c^2(\mu_5(2\mu_3^2 - \mu_7) + 3\mu_2^2(6\mu_3^2 + \mu_7))) \\
\Pi_{3,10} &= -2c^{-17}(39c^{14}\mu_2 + 72c^{13}\mu_3 - 648c^{10}\mu_2\mu_3 - 72c^7\mu_3(9\mu_2^2 - 20\mu_5) \\
&\quad - 1296c^4\mu_2\mu_3\mu_5 - 6c^{11}(3\mu_2^2 + 4\mu_5) - 54c^8(4\mu_2^3 - 13\mu_2\mu_5) \\
&\quad - 6c^2\mu_2(5 + 648\mu_2^4 - 108\mu_2^2\mu_5 - 54\mu_5^2) - 12c\mu_3(1 + 648\mu_2^4 \\
&\quad - 540\mu_2^2\mu_5 - 18\mu_5^2) + 216c^6\mu_2(2\mu_3^2 - \mu_7) + 6c^9(-190\mu_3^2 + \mu_7) \\
&\quad + 216\mu_2\mu_5(2\mu_3^2 + \mu_7) + 2c^5(-1 + 648\mu_2^4 + 936\mu_3^3 + 324\mu_2^2\mu_5 \\
&\quad - 270\mu_5^2 - 180\mu_3\mu_7) - 216c^3(\mu_5(6\mu_3^2 - \mu_7) + 2\mu_2^2(14\mu_3^2 + \mu_7))) \\
\Pi_{3,11} &= -6c^{-18}(14c^{15}\mu_2 + 26c^{14}\mu_3 - 288c^{11}\mu_2\mu_3 - 108c^8\mu_3(8\mu_2^2 - 5\mu_5) - \\
&\quad 1296c^5\mu_2\mu_3(4\mu_2^2 - \mu_5) - 19c^{12}\mu_5 + c^9(-72\mu_2^3 + 264\mu_2\mu_5) \\
&\quad - 12c^2\mu_3(1 + 216\mu_2^4 + 72\mu_2^2\mu_5 + 6\mu_5^2) + 2(\mu_5 + 18\mu_5^3) \\
&\quad - 144c\mu_2(3\mu_2^2 - \mu_5)(2\mu_3^2 + \mu_7) - 48c^7\mu_2(\mu_3^2 + 2\mu_7) \\
&\quad + c^{10}(-400\mu_3^2 + 4\mu_7) + 2c^3\mu_2(-5 + 648\mu_2^4 - 720\mu_3^3 - 324\mu_2^2\mu_5 \\
&\quad - 162\mu_5^2 - 360\mu_3\mu_7) - 12c^6(108\mu_2^4 - 48\mu_3^3 - 99\mu_2^2\mu_5 + 8\mu_5^2 \\
&\quad + 12\mu_3\mu_7) - 36c^4(\mu_5(10\mu_3^2 - \mu_7) + 4\mu_2^2(22\mu_3^2 + 5\mu_7))) \\
\Pi_{3,12} &= c^{-18}(c^{18} - 84c^{15}\mu_2 - 180c^{14}\mu_3 + 1368c^{11}\mu_2\mu_3 + 144c^8\mu_3(21\mu_2^2 - 31\mu_5) \\
&\quad + 864c^5\mu_2\mu_3(33\mu_2^2 - 4\mu_5) + c^{12}(-288\mu_2^2 + 198\mu_5) - 72c^9(9\mu_2^3 + 20\mu_2\mu_5) \\
&\quad + 36(\mu_5(-1 + 6\mu_5^2) + \mu_2^2(2 + 36\mu_5^2)) + 96c^{10}(25\mu_3^2 - \mu_7) \\
&\quad + 216c^7\mu_2(2\mu_3^2 + \mu_7) - 2592c\mu_2(\mu_2^2 - \mu_5)(2\mu_3^2 + \mu_7) \\
&\quad - 24c^3\mu_2(-1 + 972\mu_2^4 - 432\mu_3^3 - 864\mu_2^2\mu_5 + 306\mu_5^2 - 216\mu_3\mu_7) \\
&\quad + 2c^6(-1 + 8424\mu_2^4 - 864\mu_3^3 - 4968\mu_2^2\mu_5 + 1386\mu_5^2 + 864\mu_3\mu_7) \\
&\quad + 72c^2(24\mu_3^4 + \mu_3(1 - 648\mu_2^4 + 360\mu_2^2\mu_5 - 18\mu_5^2) + 24\mu_3^2\mu_7 + 6\mu_7^2) \\
&\quad + 432c^4(-\mu_5(4\mu_3^2 + 5\mu_7) + \mu_2^2(66\mu_3^2 + 9\mu_7))) \\
\Pi_{3,13} &= c^{-18}(c^{18} - 96c^{14}\mu_3 - 288c^{11}\mu_2\mu_3 + 864c^5\mu_2\mu_3(6\mu_2^2 - 7\mu_5) \\
&\quad - 1296c^8\mu_3(3\mu_2^2 + \mu_5) + c^{12}(-36\mu_2^2 + 60\mu_5) - 36c^9(24\mu_2^3 - 5\mu_2\mu_5) \\
&\quad - 24c^2\mu_3(1 + 1944\mu_2^4 - 432\mu_2^2\mu_5 + 126\mu_5^2) + 24(\mu_5 + 18\mu_5^3) \\
&\quad + \mu_2^2(3 - 162\mu_5^2)) + 288c^7\mu_2(10\mu_3^2 - \mu_7) + 12c^{10}(118\mu_3^2 - \mu_7) \\
&\quad - 432c\mu_2(12\mu_2^2 + \mu_5)(2\mu_3^2 + \mu_7) - 24c^3\mu_2(-1 + 648\mu_2^4 - 288\mu_3^3 \\
&\quad - 486\mu_2^2\mu_5 - 144\mu_3\mu_7) + 2c^6(1 + 3240\mu_2^4 - 2448\mu_3^3 + 324\mu_2^2\mu_5 \\
&\quad - 126\mu_5^2 + 72\mu_3\mu_7) + 144c^4(6\mu_2^2(22\mu_3^2 - \mu_7) + \mu_5(50\mu_3^2 + \mu_7)))
\end{aligned}$$

## F $E_8$ geometry

### F.1 LMO side: minimal orbit

The orbit consists of 240 30-dimensional vectors with entries  $-2, -1, 0, 1, 2$ . If  $w$  is in the orbit, so does  $-w$  and its shift  $\varepsilon(w) = (w_{j+1 \bmod a})_j$ . Below we give an element of  $\{\pm w, \pm \varepsilon(w), \dots\}$  that has



$w_0 \neq 0$  and  $n_0(w) \geq 0$ , and indicate the size  $l$  of this sub-orbit. The vectors are compactly encoded in  $\underline{w}(t) = \sum_{k=0}^{29} w_k t^k$ .

$n_0$	$l$	$\underline{w}(t)$
$\pm 6$	$2 \cdot 5$	$1 + t^5 + t^{10} + t^{15} + t^{20} + t^{25}$
$\pm 5$	$2 \cdot 6$	$-1 + t + t^5 - t^6 + t^7 + t^{11} - t^{12} + t^{13} + t^{17} - t^{18} + t^{19} + t^{23} - t^{24} + t^{25} + t^{29}$
$\pm 4$	$2 \cdot 15$	$-1 + t + t^4 - t^5 + t^7 + t^{13} - t^{15} + t^{16} + t^{19} - t^{20} + t^{22} + t^{28}$
$\pm 3$	$2 \cdot 10$	$-1 + t + t^3 - t^4 + t^7 - t^{10} + t^{11} + t^{13} - t^{14} + t^{17} - t^{20} + t^{21} + t^{23} - t^{24} + t^{27}$
$\pm 3$	$2 \cdot 10$	$1 - t + t^2 - t^3 + t^4 + t^{10} - t^{11} + t^{12} - t^{13} + t^{14} + t^{20} - t^{21} + t^{22} - t^{23} + t^{24}$
$\pm 2$	$2 \cdot 15$	$-1 + t + t^2 - t^3 + t^7 - t^9 + t^{11} - t^{15} + t^{16} + t^{17} - t^{18} + t^{22} - t^{24} + t^{26}$
$\pm 2$	$2 \cdot 15$	$1 - t + t^3 - t^4 + t^5 - t^6 + t^8 + t^{15} - t^{16} + t^{18} - t^{19} + t^{20} - t^{22} + t^{23}$
$\pm 1$	$2 \cdot 30$	$2 - t + t^6 - t^7 + t^{10} - t^{11} + t^{12} - t^{13} + t^{15} - t^{17} + t^{18} - t^{19} + t^{20} - t^{23} + t^{24} - t^{29}$
0	$2 \cdot 5$	$-1 + t^4 - t^5 + t^9 - t^{10} + t^{14} - t^{15} + t^{19} - t^{20} + t^{24} - t^{25} + t^{29}$
0	$2 \cdot 5$	$1 - t^2 + t^5 - t^7 + t^{10} - t^{12} + t^{15} - t^{17} + t^{20} - t^{22} + t^{25} - t^{27}$
0	$2 \cdot 3$	$1 - t + t^3 - t^4 + t^6 - t^7 + t^9 - t^{10} + t^{12} - t^{13} + t^{15} - t^{16} + t^{18} - t^{19} + t^{21} - t^{22} + t^{24} - t^{25} + t^{26} - t^{28}$
0	2	$\sum_{j=0}^{29} (-1)^j t^j$

## F.2 LMO side: Newton polygon

The boundary (and coefficients therein) of the Newton polygon of the full curve  $\mathcal{P}_{E_8}^{\text{LMO}}(X, Y)$  is the same as the one of the polynomial computed in terms of the minimal orbit data:

$$C \prod_{i=1}^{240} (Y - (-cx)^{n_0(w[i])} \zeta_{30}^{n_1(w[i])}), \quad X = x^{30}, \quad c = e^{\lambda/1800}. \quad (\text{F.1})$$

where the constant  $C$  is fixed so that the monomial  $X^9 Y^0$  appears with coefficient 1. The result is:

$$\begin{aligned} \mathcal{P}_{E_8}^{\text{LMO}}(X, Y) = & -c^{-240} (1 + X^{18}) Y^{106} (Y + 1)^2 (Y^2 + Y + 1)^3 (Y^4 + Y^3 + Y^2 + Y + 1) \\ & + c^{210} (X + X^{17}) (Y^{76} + \dots + Y^{164}) + 2c^{180} (X^2 + X^{16}) (Y^{61} + \dots + Y^{179}) \\ & + c^{150} (X^3 + X^{15}) (Y^{46} + \dots + Y^{194}) - 2c^{120} (X^4 + X^{14}) (Y^{36} + \dots + Y^{204}) \\ & + c^{90} (X^5 + X^{13}) (Y^{26} + \dots + Y^{214}) - c^{60} (X^7 + X^{11}) (Y^{11} + \dots + Y^{229}) \\ & + c^{30} (X^8 + X^{10}) (Y^5 + \dots + Y^{235}) + X^9 (1 + Y^{240}), \end{aligned}$$

where the  $\dots$  lie in the interior of the polygon.

## References

- [AENV14] M. Aganagic, T. Ekhholm, L. Ng, and C. Vafa, *Topological strings, D-model, and knot contact homology*, Adv. Theor. Math. Phys. **18** (2014), 827–956, hep-th/1304.5778.
- [AKMV04] M. Aganagic, A. Klemm, M. Mariño, and C. Vafa, *Matrix model as a mirror of Chern-Simons theory*, JHEP **0402** (2004), 010, hep-th/0211098.
- [AMMM14] A. Alexandrov, A. Mironov, A. Morozov, and An. Morozov, *Towards matrix model representation of HOMFLY polynomials*, JETP Lett. **100** (2014), no. 4, 271–278, hep-th/1407.3754.
- [AV00] M. Aganagic and C. Vafa, *Mirror symmetry, D-branes and counting holomorphic discs*, hep-th/0012041.
- [BC11] A. Brini and R. Cavalieri, *Open orbifold Gromov-Witten invariants of  $[\mathbb{C}^3/\mathbb{Z}_n]$ : localization and mirror symmetry*, Selecta Math. (N.S.) **17** (2011), no. 4, 879–933, math.AG/1007.0934.
- [BCR13] A. Brini, R. Cavalieri, and D. Ross, *Crepant resolutions and open strings*, math.AG/1309.4438.
- [BE14] G. Borot and B. Eynard, *Spectral curves, root systems, and application to  $SU(N)$  Chern-Simons theory on Seifert spaces*, math-ph/1407.4500.
- [Bea13] C. Beasley, *Localization for Wilson Loops in Chern-Simons Theory*, Adv. Theor. Math. Phys. **17** (2013), 1–240, hep-th/0911.2687.
- [BEM12] A. Brini, B. Eynard, and M. Mariño, *Torus knots and mirror symmetry*, Annales Henri Poincaré **13** (2012), no. 8, 1873–1910, hep-th/1105.2012.
- [BEO15] G. Borot, B. Eynard, and N. Orantin, *Abstract loop equations, topological recursion, and applications*, Commun. Number Theory and Physics (2015), math-ph/1303.5808.
- [BFM04] V. Bouchard, B. Florea, and M. Mariño, *Counting higher genus curves with crosscaps in Calabi-Yau orientifolds*, JHEP **0412** (2004), 035, hep-th/0405083.
- [BFM05] ———, *Topological open string amplitudes on orientifolds*, JHEP **0502** (2005), 002, hep-th/0411227.
- [BG08] J. Bryan and A. Gholampour, *Root systems and the quantum cohomology of ADE resolutions*, Algebra Number Theory **2** (2008), no. 4, 369–390.
- [BG09] J. Bryan and T. Graber, *The crepant resolution conjecture*, Proc. Sympos. Pure Math. **80** (2009), 23–42.
- [BGK15] G. Borot, A. Guionnet, and K. Kozłowski, *Large- $N$  asymptotic expansion for mean field models with Coulomb gas interaction*, Int. Math. Res. Not. (2015), math-ph/1312.6664.

- [BGST10] A. Brini, L. Griguolo, D. Seminara, and A. Tanzini, *Chern-Simons theory on  $L(p, q)$  lens spaces and Gopakumar-Vafa duality*, J. Geom. Phys. **60** (2010), no. 3, 417–429, math.MP/0809.1610.
- [BIS<sup>+</sup>97] A. Brandhuber, N. Itzhaki, J. Sonnenschein, S. Theisen, and S. Yankielowicz, *On the M-theory approach to (compactified) 5d theories*, hep-th/9709010.
- [BKMP09] V. Bouchard, A. Klemm, M. Mariño, and S. Pasquetti, *Remodeling the B-model*, Commun. Math. Phys. **287** (2009), 117–178, hep-th/0709.1453.
- [BN06] D. Bar-Natan, *Finite type invariants*, Encyclopedia of Mathematical Physics (G.L. Naber J.-P. Francoise and S.T. Tsou, eds.), vol. 2, Oxford, 2006, math.GT/0408182, p. 340.
- [BNL04] D. Bar-Natan and R. Lawrence, *A rational surgery formula for the LMO invariant*, Israel J. Math. **140** (2004), 29–60, math.GT/0007045.
- [Bor14] G. Borot, *Formal multidimensional integrals, stuffed maps, and topological recursion*, Annales Institut Poincaré - D **1** (2014), no. 2, 225–264, math-ph/1307.4957.
- [Bri12] A. Brini, *Open topological strings and integrable hierarchies: Remodeling the A-model*, Commun.Math.Phys. **312** (2012), 735–780, 1102.0281.
- [Bri15] A. Brini, *Mirror symmetry and the higher genus crepant resolution conjecture for simple singularities*, work in progress (2015).
- [BT06] M. Blau and G. Thompson, *Chern-Simons theory on  $S^1$ -bundles: abelianisation and  $q$ -deformed Yang-Mills theory*, JHEP **0605** (2006), no. 003, hep-th/0601068.
- [BT13] ———, *Chern-Simons theory on Seifert manifolds*, JHEP **33** (2013), hep-th/1306.3381.
- [BW05] C. Beasley and E. Witten, *Non-abelian localization for Chern-Simons theory*, J. Diff. Geom. **70** (2005), 183–323, hep-th/0503126.
- [DBOSS14] P. Dunin-Barkowski, N. Orantin, S. Shadrin, and L. Spitz, *Identification of the Givental formula with the spectral curve topological recursion procedure*, Commun. Math. Phys. **328** (2014), 669–700, math-ph/1211.4021.
- [DFM03] D-E. Diaconescu, B. Florea, and A. Misra, *Orientifolds, unoriented instantons and localization*, JHEP **0307** (2003), 041, hep-th/0305021.
- [DGG14] T. Dimofte, D. Gaiotto, and S. Gukov, *Gauge theories labelled by three-manifolds*, Commun. Math. Phys. **325** (2014), no. 2, 367–419, hep-th/1108.4389.
- [DSV13] D.E. Diaconescu, V. Shende, and C. Vafa, *Large  $N$  duality, lagrangian cycles, and algebraic knots*, Commun. Math. Phys. **319** (2013), 813–863, hep-th/1111.6533.
- [DT98] S. K. Donaldson and R. P. Thomas, *Gauge theory in higher dimensions*, The geometric universe (Oxford, 1996), Oxford Univ. Press, Oxford, 1998, pp. 31–47.

- [ELG15] P. Etingof, I. Losev, and E. Gorsky, *Representations of rational Cherednik algebras with minimal support and torus knots*, Adv. Math. **277** (2015), 124–180, math.RT/1304.3412.
- [EO07] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, Commun. Number Theory and Physics **1** (2007), no. 2, math-ph/0702045.
- [EO15] ———, *Computation of open Gromov-Witten invariants for toric Calabi-Yau 3-folds by topological recursion, a proof of the BKMP conjecture*, Commun. Math. Phys. **337** (2015), no. 2, 483–567, math-ph/1205.1103.
- [EWY01] T. Eguchi, N.P. Warner, and S.-K. Yang, *ADE singularities and coset models*, Nucl. Phys. B **607** (2001), 3–37, hep-th/0105194.
- [FM97] V. Fock and A. Marshakov, *A note on quantum groups and relativistic Toda theory*, Nucl. Phys. Proc. Suppl. B **56** (1997), 208–214.
- [FM14] V.V. Fock and A. Marshakov, *Loop groups, clusters, dimers and integrable systems*, math.AG/1401.1606.
- [FP03] C. Faber and R. Pandharipande, *Hodge integrals, partition matrices, and the  $\lambda_g$  conjecture*, Ann. of Math. (2) **157** (2003), no. 1, 97–124, math.AG/9810173.
- [GNY09] L. Göttsche, H. Nakajima, and K. Yoshioka, *K-theoretic Donaldson invariants via instanton counting*, Pure Appl. Math. Quart. **5** (2009), 1029–1111, math/0611945.
- [GORS14] E. Gorsky, A. Oblomkov, J. Rasmussen, and V. Shende, *Torus knots and the rational DAHA*, Duke Math. J. **163** (2014), no. 14, 2709–2794, math.RT/1207.4523.
- [GP99] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), no. 2, 487–518.
- [GP00] M. Geck and G. Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, London Mathematical Society Monographs, no. 21, Oxford Science Publications, 2000.
- [GP15] S. Gukov and D. Pei, *Equivariant Verlinde formula from fivebranes and vortices*, hep-th/1501.01310.
- [GV99] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, Adv. Theor. Math. Phys. **3** (1999), 1415–1443, hep-th/9811131.
- [Han01] S.K. Hansen, *Reshetikhin-Turaev invariants of Seifert 3-manifolds and a rational surgery formula*, Algebr. Geom. Topol. **1** (2001), 627–686, math.GT/0111057.
- [HIV00] K. Hori, A. Iqbal, and C. Vafa, *D-branes and mirror symmetry*, hep-th/0005247.
- [HRT01] R.B. Howlett, L.J. Rylands, and D.E. Taylor, *Matrix generators for exceptional groups of Lie type*, J. Symbolic Comput. **31** (2001), no. 4, 429–445.

- [HY09] N. Halmagyi and V. Yasnov, *The spectral curve of the lens space matrix model*, JHEP **0911** (2009), 104, hep-th/0311117.
- [JKS14] H. Jockers, A. Klemm, and M. Soroush, *Torus knots and the topological vertex*, Lett. Math. Phys. **104** (2014), 953–989.
- [K11] J. Källén, *Cohomological localization of Chern-Simons theory*, JHEP (2011), no. 8, 1–32, hep-th/1104.5353v2.
- [KKL<sup>+</sup>96] S. Kachru, A. Klemm, W. Lerche, P. Mayr, and C. Vafa, *Nonperturbative results on the point particle limit of  $N=2$  heterotic string compactifications*, Nucl. Phys. B **459** (1996), 537–558, hep-th/9508155.
- [KKV97] S.H. Katz, A. Klemm, and C. Vafa, *Geometric engineering of quantum field theories*, Nucl. Phys. B **497** (1997), 173–195, hep-th/9609239.
- [KL02] S.H. Katz and C.-C.M. Liu, *Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc*, Geom. Topol. Monogr. **8** (2002), 1–47, math.AG/0103074.
- [KLM<sup>+</sup>96] A. Klemm, W. Lerche, P. Mayr, C. Vafa, and N.P. Warner, *Selfdual strings and  $N=2$  supersymmetric field theory*, Nucl. Phys. B **477** (1996), 746–766, hep-th/9604034.
- [KM15] O. Kruglinskaya and A. Marshakov, *On Lie groups and Toda lattices*, J. Phys. A **48** (2015), no. 12, 125201, hep-th/1404.6507.
- [Kon94] M. Kontsevich, *Enumeration of rational curves via torus actions*, Progress in Mathematics, vol. 129, pp. 335–368, Birkhäuser, Boston, 1994, hep-th/9405035.
- [KS97] Y. Kosmann-Schwarzbach, *Lie bialgebras, Poisson Lie groups and dressing transformations*, Integrability of nonlinear systems (Pondicherry, 1996), Lecture Notes in Phys., vol. 495, Springer, Berlin, 1997, pp. 104–170.
- [KWY10] A. Kapustin, B. Willett, and I. Yaakov, *Exact results for Wilson loops in superconformal Chern-Simons theories with matter*, JHEP (2010), no. 1003:089, hep-th/0909.4559.
- [LMO98] T.Q.T. Lê, J. Murakami, and T. Ohtsuki, *On a universal perturbative invariant of 3-manifolds*, Topology **37** (1998), no. 3, 539–574, q-alg/9512002.
- [LMV00] J.M.F. Labastida, M. Mariño, and C. Vafa, *Knots, links and branes at large  $N$* , JHEP **0011** (2000), 007, hep-th/0010102.
- [LN98] A.E. Lawrence and N. Nekrasov, *Instanton sums and five-dimensional gauge theories*, Nucl. Phys. **B513** (1998), 239–265, hep-th/9706025.
- [LR99] R. Lawrence and L. Rozansky, *Witten-Reshetikhin-Turaev invariants of Seifert manifolds*, Commun. Math. Phys. **205** (1999), 287–314.
- [LW98] W. Lerche and N.P. Warner, *Exceptional SW geometry from ALE fibrations*, Phys. Lett. B **423** (1998), 79–86, hep-th/9608183.

- [Mar04] M. Mariño, *Chern-Simons theory, matrix integrals, and perturbative three-manifold invariants*, Commun. Math. Phys. **253** (2004), 25–49, hep-th/0207096.
- [Mar05] ———, *Chern-Simons theory and topological strings*, Rev. Mod. Phys. **77** (2005), 675–720, hep-th/0406005.
- [Mar13] A. Marshakov, *Lie groups, cluster variables and integrable systems*, J. Geom. Phys. **67** (2013), 16–36, hep-th/1207.1869.
- [McK80] J. McKay, *Graphs, singularities and finite groups*, Proc. Symp. Pure Math. AMS **37** (1980), 183–186.
- [Mil57] J. Milnor, *Groups which act on  $\mathbb{S}^n$  without fixed points*, Amer. J. Math. **79** (1957), 623–630.
- [MMM<sup>+</sup>15] A. Mironov, A. Morozov, An. Morozov, P. Ramadevi, and V. K. Singh, *Colored HOM-FLY polynomials of knots presented as double fat diagrams*, hep-th/1504.00371.
- [MOOP08] D. Maulik, A. Oblomkov, A. Okounkov, and R. Pandharipande, *Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds*, Inventiones Mathematicae (2008), 1–45, math.AG/0809.3976.
- [MW96] E.J. Martinec and N.P. Warner, *Integrable systems and supersymmetric gauge theory*, Nucl. Phys. B **459** (1996), 97–112, hep-th/9509161.
- [Nek98] N. Nekrasov, *Five dimensional gauge theories and relativistic integrable systems*, Nucl. Phys. B **531** (1998), 323–344, hep-th/9609219.
- [Orl72] P. Orlik, *Seifert manifolds*, Lecture Notes in Mathematics, vol. 291, Springer, Berlin, 1972.
- [OV00] H. Ooguri and C. Vafa, *Knot invariants and topological strings*, Nucl. Phys. **B577** (2000), 419–438, hep-th/9912123.
- [Rei] M. Reid, *The Du Val singularities  $A_n$ ,  $D_n$ ,  $E_6$ ,  $E_7$ ,  $E_8$* , <http://homepages.warwick.ac.uk/~masda/surf/more/DuVal.pdf>.
- [RT90] N.Yu. Reshetikhin and V.G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Commun. Math. Phys. **127** (1990), 1–26.
- [RT91] ———, *Invariants of three manifolds via link polynomials and quantum groups*, Invent. Math. **103** (1991), 547–597.
- [Rui90] S.N.M. Ruijsenaars, *Relativistic Toda systems*, Commun. Math. Phys. **133** (1990), no. 2, 217–247.
- [Sei80] H. Seifert, *Topology of 3-dimensional fibered spaces*, Academic Press, 1980.
- [SV00] S. Sinha and C. Vafa, *SO and Sp Chern-Simons at large N*, hep-th/0012136.

- [tH74] G. 't Hooft, *A planar diagram theory for strong interactions*, Nucl. Phys. B **72** (1974), 461–473.
- [TZ08] S. Tomoda and P. Zvengrowski, *Remarks on the cohomology of finite fundamental groups of 3-manifolds*, Geometry and Topology Monographs **14** (2008), 519–556.
- [Wal92] K. Walker, *An extension of Casson's invariant*, vol. 126, Princeton University Press, Princeton, NJ, 1992.
- [Wil13] H. Williams, *Double Bruhat cells in Kac-Moody groups and integrable systems*, Lett. Math. Phys. **103** (2013), 389–419, math.QA/1204.0601.
- [Wit89] E. Witten, *Quantum field theory and the Jones polynomial*, Commun. Math. Phys. **121** (1989), 351.
- [Wit95] E. Witten, *Chern-Simons gauge theory as a string theory*, Prog. Math. **133** (1995), 637–678, hep-th/9207094.
- [Zas93] E. Zaslow, *Topological orbifold models and quantum cohomology rings*, Commun. Math. Phys. **156** (1993), 301–332, hep-th/9211119.
- [Zim11] B.P. Zimmermann, *On finite groups acting on spheres and finite subgroups of orthogonal groups*, math.GT/1108.2602.